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Inégalités de concentrations non commutatives dans un cadre de dépendance

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Bernstein inequality
Scalar independent case

Let $X_1, \ldots, X_n$ be independent random variables such that

$$\mathbb{E}X_k = 0\,, \quad \mathbb{E}X_k^2 = \sigma_k^2 \quad \text{and} \quad \sup_k |X_k| < 1 \text{ a.s.}$$

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For any $x > 0$,

$$\mathbb{P}\Big(\sum_{k=1}^{n} X_k > x\Big) \leqslant \exp\Big(-\frac{x^2}{2\,nV_n + 2x}\Big)\,,$$

where $V_n = \frac{1}{n}\sum_{k=1}^{n} \sigma_k^2$.

Matrix Setting

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be $d \times d$ *centered* Hermitian random matrices.

What can be said about

$$\mathbb{P}\Big(\lambda_{\max}\big(\sum_{k=1}^{n} \mathbf{X}_k\big) \geqslant x\Big) \leqslant \mathbf{?}$$

Independent Matrix Case

Theorem (Ahlswede and Winter '02, Tropp '12)

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be $d \times d$ independent Hermitian random matrices. Assume that each matrix satisfies

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Then for any $x > 0$,

$$\mathbb{P}\Big(\lambda_{\max}\big(\sum_{k=1}^{n} \mathbf{X}_k\big) \geq x\Big) \leq d \cdot \exp\Big(-\frac{x^2/2}{n\sigma^2 + x/3}\Big),$$

where $\sigma^2 := \frac{1}{n}\lambda_{\max}\big(\sum_{k=1}^{n} \mathbb{E}\mathbf{X}_k^2\big)$.

Applications?!

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Question: What is the minimal order of n so that

$$\|\frac{1}{n} \sum_{k=1}^{n} Y_k Y_k^t - Id\| < \epsilon?$$

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Bernstein's inequality then yields

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Conclusion: $n \sim d\log d$ copies are sufficient.

Dependent Matrix Case

- The β-mixing coefficient between two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\beta(\mathcal{A}, \mathcal{B}) = \frac{1}{2} \sup \left\{ \sum_{i \in I} \sum_{j \in J} \left| \mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j) \right| \right\},$$

where the supremum is taken over all finite partitions $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ that are respectively $\mathcal{A}$ and $\mathcal{B}$ measurable.

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- The sequence $(\beta_k)_k$ associated with $(\mathbf{X}_k)_k$ is defined by

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- $\beta_k \leq \mathrm{e}^{-ck}$ for some positive constant c.

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Then for any $x > 0$,

$$\mathbb{P}\Big(\lambda_{\max}\Big(\sum_{k=1}^{n} \mathbf{X}_k\Big) \geq x\Big) \leq d \exp\Big(-\frac{Cx^2}{nv^2 + xc^{-1}(\log n)^2}\Big),$$

where C is a universal constant and v^2 is given by

$$v^2 = \sup_{J\subseteq\{1,\dots,n\}} \frac{1}{\mathrm{Card} J} \lambda_{\max}\Big(\sum_{k,\ell\in J} \mathrm{Cov}(\mathbf{X}_k, \mathbf{X}_\ell)\Big).$$

Consequence

Let **A** be a $d \times n$ random matrix such that:

- Y_i= Columns of A are isotropic i.e. $\mathbb{E} Y_i = 0$ and $\mathbb{E} Y_i Y_i^t = Id$.

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Recall that $\mathbf{AA}^T = \sum_{k=1}^n Y_k Y_k^T$ and $\lambda_k(\mathbf{AA}^T) = s_k^2(\mathbf{A})$.

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Recall that $\mathbf{A}\mathbf{A}^T = \sum_{k=1}^n Y_k Y_k^T$ and $\lambda_k(\mathbf{A}\mathbf{A}^T) = s_k^2(\mathbf{A})$.

Then with high probability

$$1 - \sqrt{d \log^3 d/n} \leq s_{\min}\Big(\frac{\mathbf{A}}{\sqrt{n}}\Big) \leq s_{\max}\Big(\frac{\mathbf{A}}{\sqrt{n}}\Big) \leq 1 + \sqrt{d \log^3 d/n}.$$

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If $d \log^3 d \ll n$ then $\frac{1}{\sqrt{n}}\mathbf{A}$ is "almost" an isometry.

Matrix Chernoff Bound

Ahlswede and Winter (2002) prove the following *matrix* Chernoff bound:

$$\mathbb{P}\Big(\lambda_{\max}\big(\sum_{k=1}^{n}\mathbf{X}_k\big) \geqslant x\Big) \leqslant \inf_{t>0}\Big\{\mathrm{e}^{-tx}\cdot\mathbb{E}\mathrm{Tr}\exp\big(t\sum_{i=1}^{n}\mathbf{X}_i\big)\Big\}$$

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Aim: Give a suitable bound for

$$L_n(t) := \mathbb{E}\,\mathrm{Tr}\exp\big(t\sum_{i=1}^{n}\mathbf{X}_i\big)$$

On the matrix exponential

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NOR convex

$$\Big(\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B}\Big)^k \not\preceq \frac{1}{2}\mathbf{A}^k + \frac{1}{2}\mathbf{B}^k \quad \text{for } k > 2\,.$$

On the Trace exponential

- The Trace exponential is increasing and convex

$$\mathbf{A} \preceq \mathbf{B} \quad \Longrightarrow \quad \operatorname{Tr}\exp(\mathbf{A}) \leq \operatorname{Tr}\exp(\mathbf{B})$$

and for any $t \in [0, 1]$,

$$\operatorname{Tr}\exp(t\mathbf{A} + (1-t)\mathbf{B}) \leq t \operatorname{Tr}\exp(\mathbf{A}) + (1-t)\operatorname{Tr}\exp(\mathbf{B})\,.$$

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- Jensen's inequality for the Trace exponential function yields

$$\mathrm{Tr}\exp(\mathbb{E}\mathbf{A}) \le \mathbb{E}\mathrm{Tr}\exp(\mathbf{A}) .$$

The Golden-Thompson Inequality

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This inequality fails for more than *two* matrices

$$\mathrm{Tr}\big(\mathrm{e}^{\mathbf{A}+\mathbf{B}+\mathbf{C}}\big) \not\leqslant \mathrm{Tr}\big(\mathrm{e}^{\mathbf{A}} \cdot \mathrm{e}^{\mathbf{B}} \cdot \mathrm{e}^{\mathbf{C}}\big)$$

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$$\mathbb{E}\mathrm{Tr}\exp\big(t\sum_{k=1}^{n}\mathbf{X}_k\big) \le \mathbb{E}\mathrm{Tr}\big(\mathrm{e}^{t\mathbf{X}_n}\cdot \mathrm{e}^{t\sum_{k=1}^{n-1}\mathbf{X}_k}\big) \qquad \text{Golden-Thompson}$$

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$$= \mathrm{Tr}\Big(\mathbb{E}(\mathrm{e}^{t\mathbf{X}_n})\cdot \mathbb{E}\big(\mathrm{e}^{t\sum_{k=1}^{n-1}\mathbf{X}_k}\big)\Big) \quad \text{Independence}$$

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$$\leq \lambda_{\max}(\mathbb{E}\mathrm{e}^{t\mathbf{X}_n})\cdot \mathbb{E}\mathrm{Tr}\left(\mathrm{e}^{t\sum_{k=1}^{n-1}\mathbf{X}_k}\right).$$

The last inequality follows from:

$$\mathbb{E}(\mathrm{e}^{t\mathbf{X}_n}) \preceq \lambda_{\max}(\mathbb{E}(\mathrm{e}^{t\mathbf{X}_n}))\cdot Id\,.$$

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$$\le \lambda_{\max}(\mathbb{E}\mathrm{e}^{t\mathbf{X}_n})\cdot\mathbb{E}\mathrm{Tr}\left(\mathrm{e}^{t\sum_{k=1}^{n-1}\mathbf{X}_k}\right).$$

Iterating this procedure:

$$\mathbb{E}\mathrm{Tr}\exp\left(t\sum_{k=1}^{n}\mathbf{X}_k\right) \leqslant d\prod_{k=1}^{n}\lambda_{\max}(\mathbb{E}\mathrm{e}^{t\mathbf{X}_k}).$$

Construction of Cantor-type sets

$\mathbf{X}_1, \ldots, \mathbf{X}_n$ geometrically β-mixing.

Aim: Bound $\mathbb{E}\mathrm{Tr} \exp\left(t \sum_{k=1}^{n} \mathbf{X}_k\right)$.

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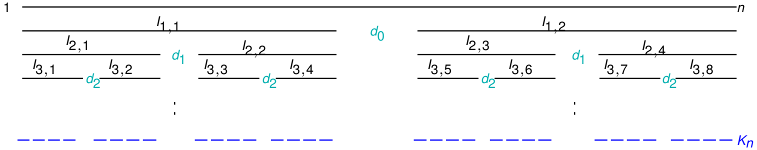

Figure: Construction of the Cantor-type set K_n

Construction of Cantor-type sets

$\mathbf{X}_1, \ldots, \mathbf{X}_n$ geometrically β-mixing.

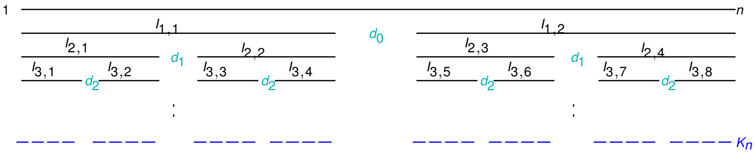

Figure: Construction of the Cantor-type set K_n

Aim: Control the Laplace transform on the Cantor set K_n

$$\mathbb{E}\mathrm{Tr}\exp\Big(t\sum_{k\in K_n}\mathbf{X}_k\Big) \le \mathbf{?}$$

Dependent matrix case
Control of the Laplace transform

- Control:

$$\mathbb{E}\mathrm{Tr}\left(e^{t\mathbf{S}_1+t\mathbf{S}_2+t\mathbf{S}_3+t\mathbf{S}_4}\right) \leq \textbf{?}$$

- Scalar Setting:

$$\mathbb{E}(e^{tS_1+tS_2+tS_3+tS_4}) \leq \mathbb{E}(e^{tS_1+tS_2}) \cdot \mathbb{E}(e^{tS_3+tS_4})(1+\epsilon(d_{2,3}))$$

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Control of the Laplace transform

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Dependent matrix case
Control of the Laplace transform

- Control:

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We shall prove instead:

$$\mathbb{E}\mathrm{Tr}\left(\mathrm{e}^{t\mathbf{S}_1+t\mathbf{S}_2+t\mathbf{S}_3+t\mathbf{S}_4}\right) \le \mathbb{E}\mathrm{Tr}\left(\mathrm{e}^{t\mathbf{S}_1^*+t\mathbf{S}_2^*+t\mathbf{S}_3^*+t\mathbf{S}_4^*}\right)(1+\epsilon(d_{1,2}, d_{2,3}, d_{3,4}))$$

Thank you for your attention!

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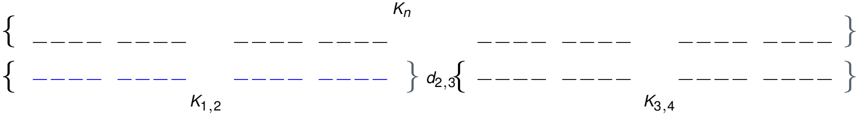


$$\sum_{k\in K_1} \mathbf{X}_k + \sum_{k\in K_2} \mathbf{X}_k + \sum_{k\in K_3} \mathbf{X}_k + \sum_{k\in K_4} \mathbf{X}_k = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 \,.$$

Step 1: Berbee's coupling allow the construction of $(\mathbf{X}_k^*)_{k\in K_3\cup K_4}$ such that

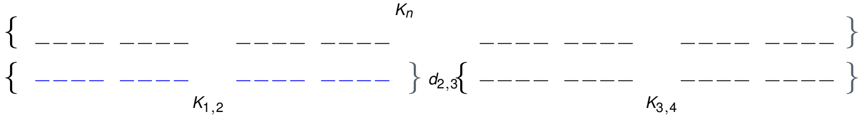
- $\mathbf{S}_3^* + \mathbf{S}_4^*$ has the same distribution as $\mathbf{S}_3 + \mathbf{S}_4$
- $\mathbf{S}_3^* + \mathbf{S}_4^*$ is independent of $\mathbf{S}_1 + \mathbf{S}_2$
- $\mathbb{P}\big(\mathbf{S}_3^* + \mathbf{S}_4^* \neq \mathbf{S}_3 + \mathbf{S}_4\big) \leqslant \beta_{d_{2,3}}$.

where

$$\mathbf{S}_3^* = \sum_{k\in K_3} \mathbf{X}_k^* \qquad \text{and} \qquad \mathbf{S}_4^* = \sum_{k\in K_4} \mathbf{X}_k^*$$

Dependent matrix case

Berbee's Coupling



$$\sum_{k\in K_1}\mathbf{X}_k+\sum_{k\in K_2}\mathbf{X}_k+\sum_{k\in K_3}\mathbf{X}_k+\sum_{k\in K_4}\mathbf{X}_k=\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3+\mathbf{S}_4\,.$$

This allows us to replace up to an error depending on $d_{2,3}$ and $|K_3|+|K_4|$

$\mathbb{E}\,\mathrm{Tr}\exp(t\mathbf{S}_1+t\mathbf{S}_2+t\mathbf{S}_3+t\mathbf{S}_4)$ by $\mathbb{E}\,\mathrm{Tr}\exp(t\mathbf{S}_1+t\mathbf{S}_2+t\mathbf{S}_3^*+t\mathbf{S}_4^*)$