Estimations de surfaces moyennes via les métamorphose de formes fonctionnelles

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Grenoble

Motivations

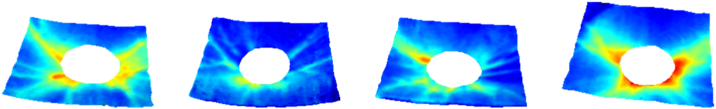
Propose a well-grounded mathematical framework to estimate and analyze variability of geometric-functional data coming from medical imaging.

- Data: for any $i = 1, \cdots, N$ the observation Y^i is in a Hilbert space $\mathcal{H}$,
- Deformations: γ^i belongs to a group $\mathcal{G}$ acting on H (*i.e* $\gamma^i \cdot Y^i \in \mathcal{H}$).
- Model:

$$Y^i = \gamma_i^* \cdot X + \varepsilon^i$$

where X is an unknown mean template and $\gamma_i^* \in \mathcal{G}$ are unknown deformations and ε^i is an error term in $\mathcal{H}$.

Functional shapes



Definition (functional shape, a.k.a. fshape)

A couple (X, f) where :

- X is a compact smooth surface of $\mathbb{R}^3$
- $f : X \to \mathbb{R}$ a function on X, with $f \in L^2(X)$, *i.e* :

$$\int_X f^2(x) d\mathcal{H}^d(x) < \infty$$

where $\mathcal{H}^d$ is the d-dimensional Hausdorff (or volume) measure.

Metamorphosis : Geometrical-functional transformations on fshape

- **Geometrical deformations :** Action of a diffeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ on a fshape (X, f)

$$\phi \cdot (X, f) = (\phi(X), f \circ \phi^{-1})$$

(*i.e* transport of the support X preserving the values of the signal.)

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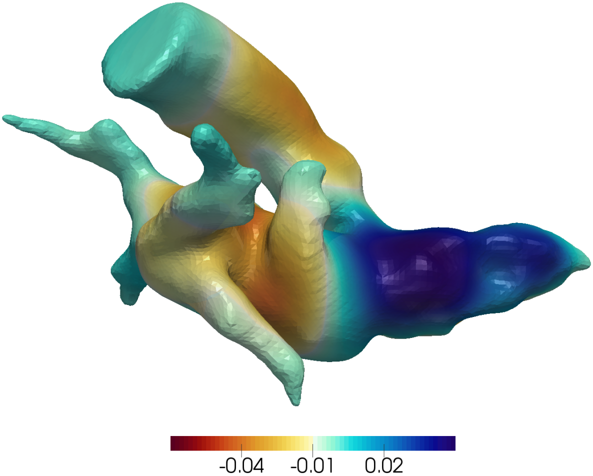
- **Functional deformations :** For a fshape $(X, f) \in \mathcal{F}$ and $(\phi, \zeta) \in G \times L^2(X)$:

$$(\phi, \zeta) \cdot (X, f) = (\phi(X), (f + \zeta) \circ \phi^{-1})$$

(*i.e* photometric variations of the signal f itself.)

Metamorphosis: an example



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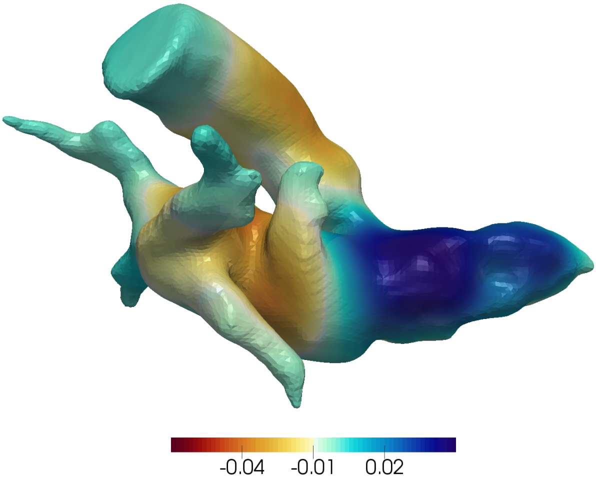

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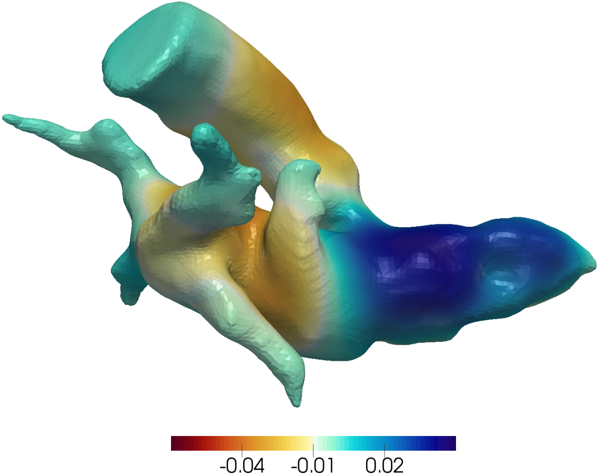

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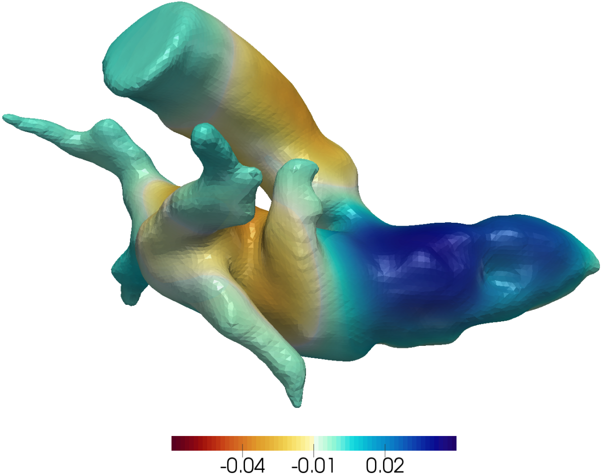

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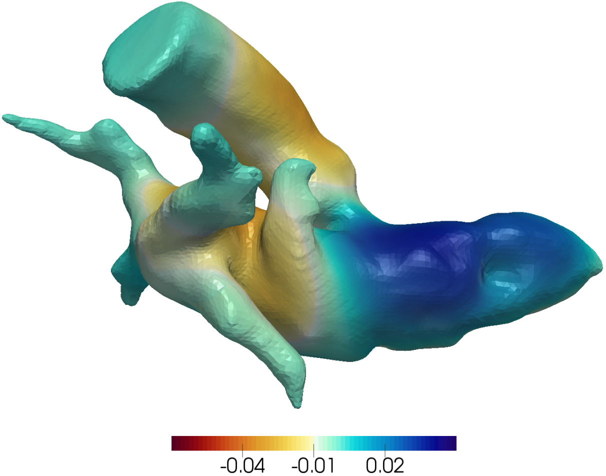

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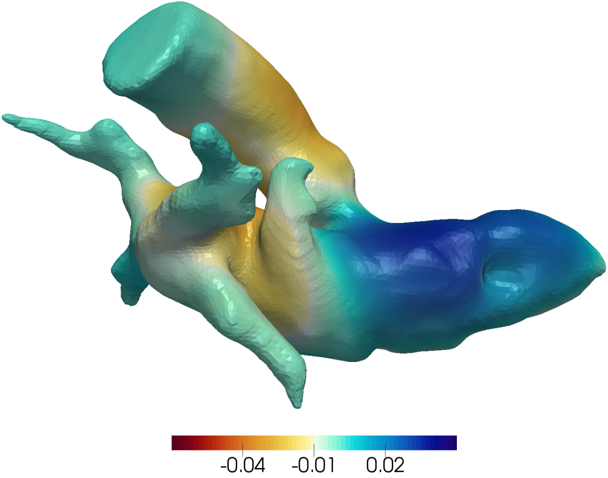

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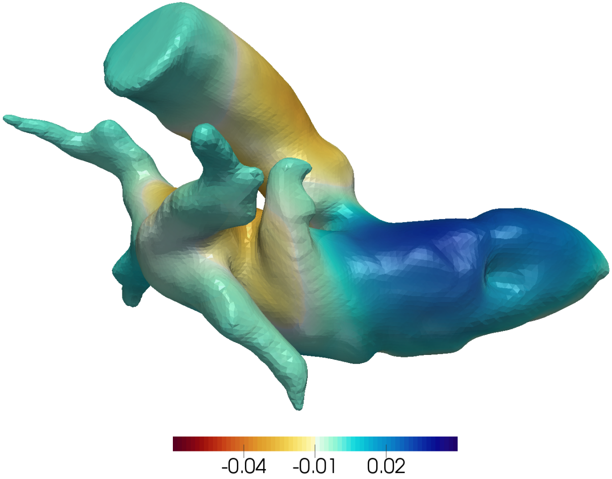

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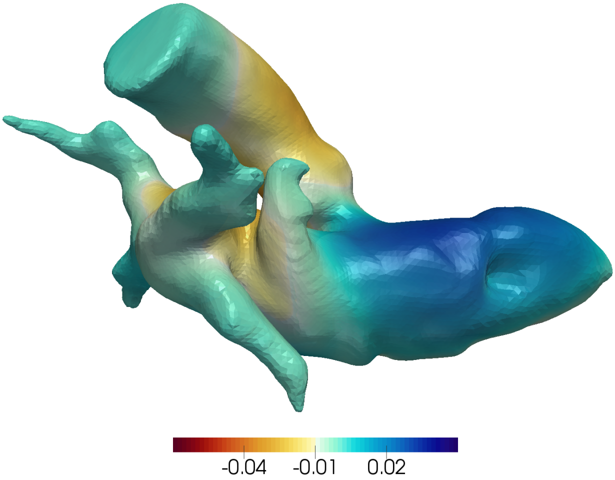

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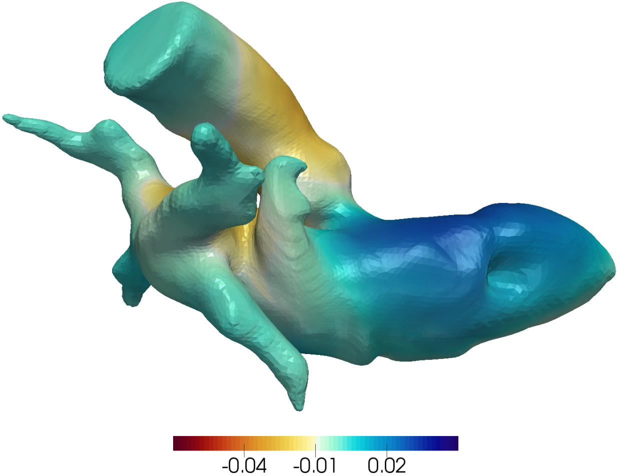

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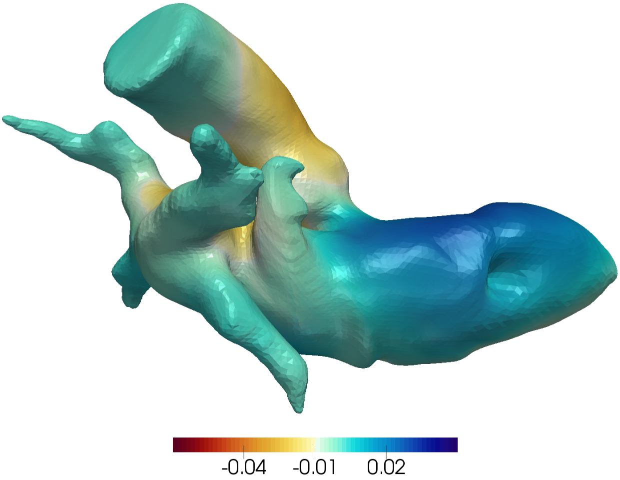

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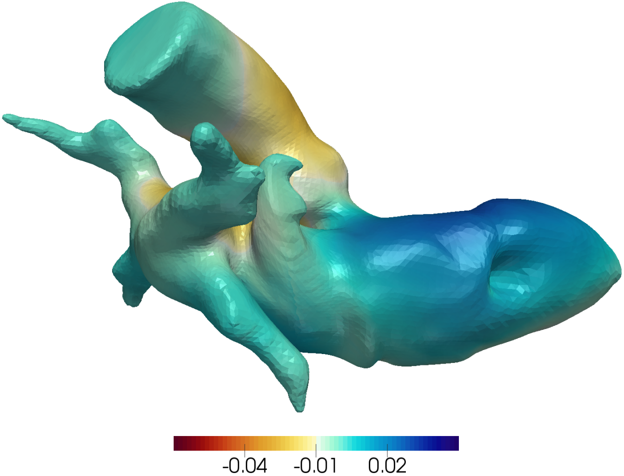

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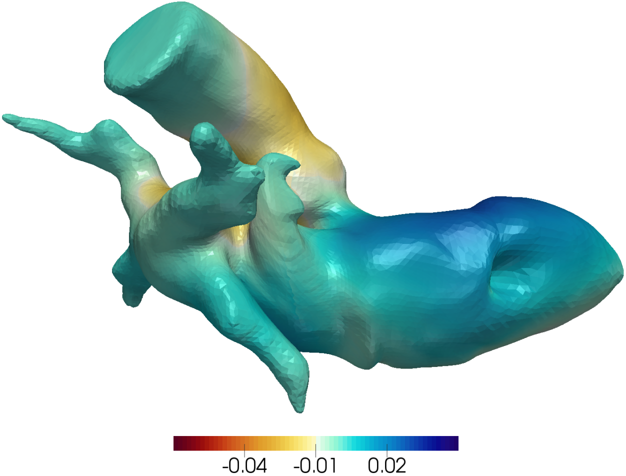

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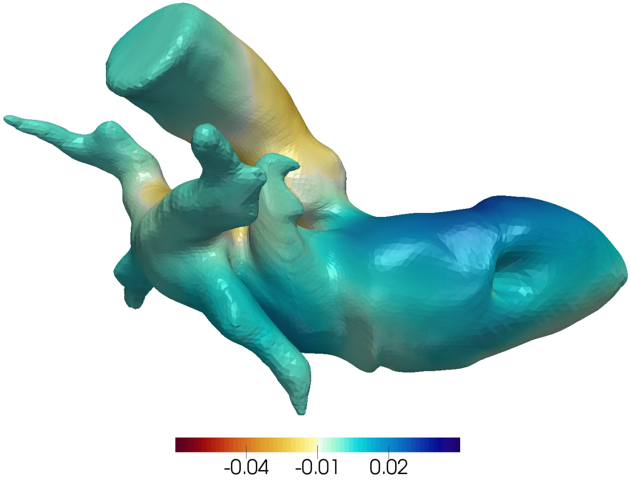

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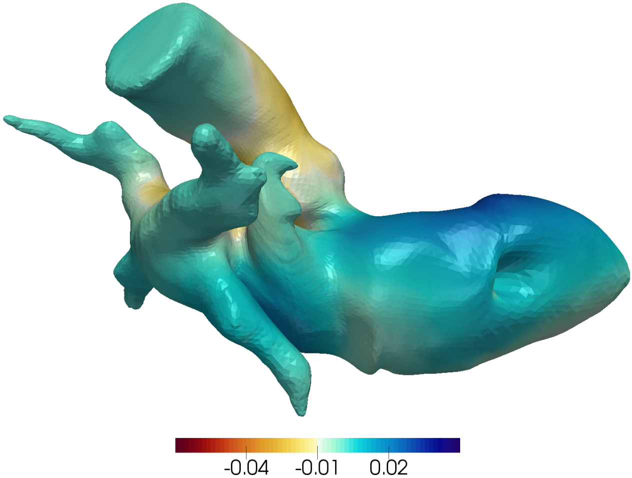

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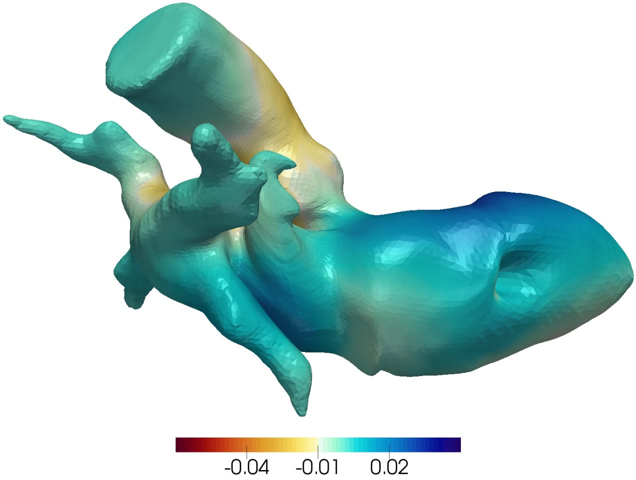

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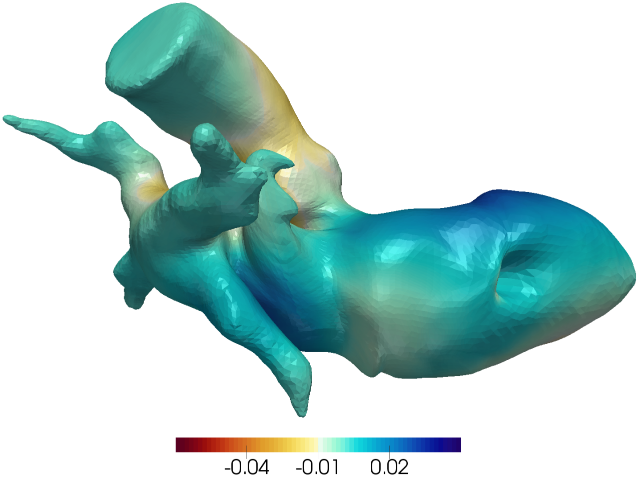

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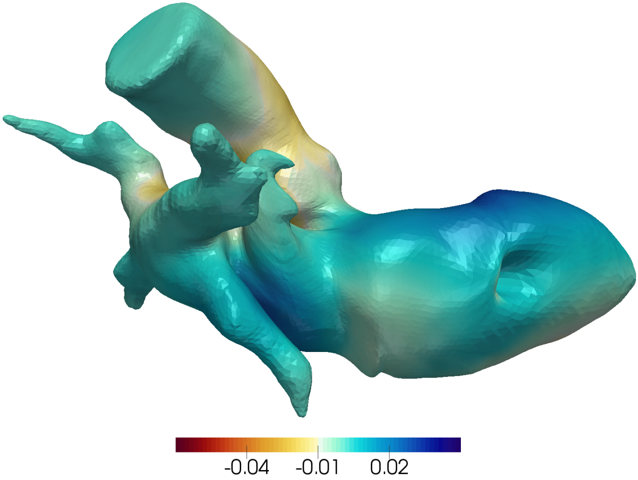

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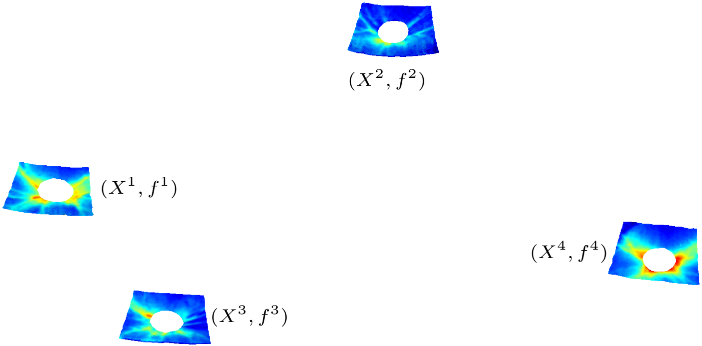
Atlas estimation for populations of fshapes

Consider a population of N fshapes $(X^i, f^i)_{i=1,\dots,N}$.



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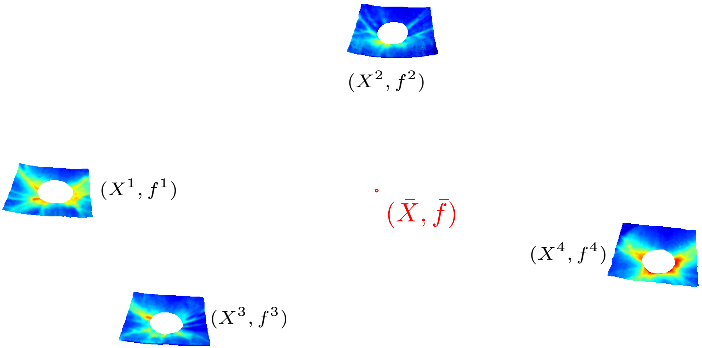

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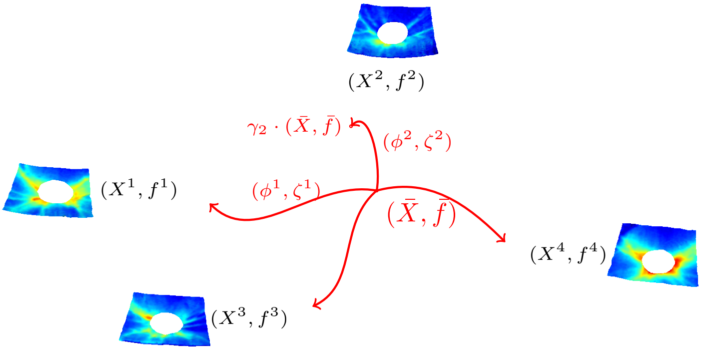


Atlas:

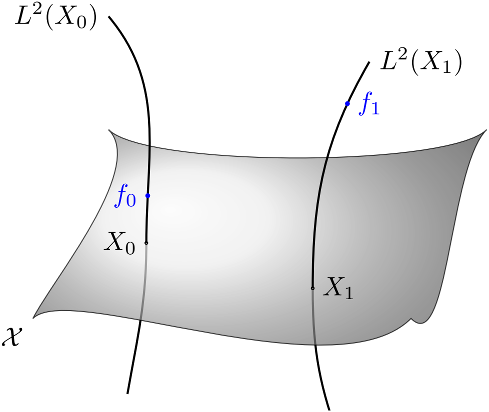
- a template fshape $(\overline{X}, \overline{f})$ that captures the average features of the population,
- N transformations $(\phi^i, \zeta^i)_{i=1,\cdots,N}$ mapping the template to each subject (*i.e* $(\phi^i, \zeta^i) \cdot (\bar{X}, \bar{f}) \approx (X^i, f^i)$).

Fshape bundle

Let $G \subset \mathrm{Diff}(\mathbb{R}^3)$ be a group of deformations. Let X_0 be a fixed submanifold of $\mathbb{R}^3$ and $\mathcal{X} = G \cdot X_0$ its orbit. We consider the space :

$$\mathcal{F} = \{ \ (X, f) \mid X \in \mathcal{X} \text{ and } f \in L^2(X) \ \}$$

which is a vector bundle of fiber $L^2(X_0)$.



Geometrical deformations : Large Deformation Diffeomorphic Metric Mapping

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- **Flow :** let $v = (v_t)_{t\in[0,1]} \in V$ be a time dependant vectors field of $\mathbb{R}^3$. Let $\phi : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$:

$$\begin{cases} \dot{\phi}_t(x) = v_t(\phi_t(x)) \\ \phi_0(x) = x. \end{cases} \qquad t \in [0,1] \text{ and } x \in \mathbb{R}^3$$

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- **Group action :** Let $L^2_V \doteq L^2([0,1],V)$. For all $v\in L^2_V$, $\phi^v_1(\cdot)$ is a $\mathcal{C}^1$-difféomorphism of $\mathbb{R}^3$. The set

$$G_V = \{\phi^v_1 : \mathbb{R}^3 \to \mathbb{R}^3, v\in L^2_V\}$$

  is a group endowed with the distance

$$d^2(\mathsf{Id},\phi) = \inf\{\|v\|^2_{L^2_V} \doteq \int_0^1 \|v_t\|^2_V dt, \dot\phi = v\circ\phi, \phi_1 = \phi\}$$

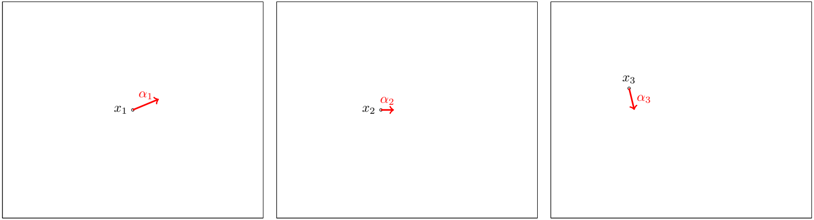
RKHS of vectors fields

- **Space of vectors fields V :** RKHS of vectors fields (smooth, vanishing at infinity). There exists a kernel $K_V : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ such that

$$\mathrm{Span}\{\delta_x^\alpha = K_V(x,\cdot)\alpha, x \in \mathbb{R}^3, \alpha \in \mathbb{R}^3\}$$

is dense in V. In practice :

$$K_V(x,y) = e^{-\frac{\|x-y\|^2}{\sigma_V^2}} Id_n.$$



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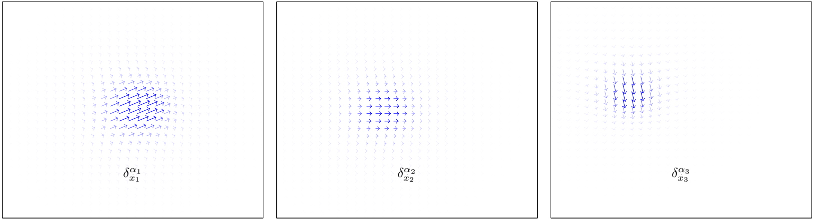

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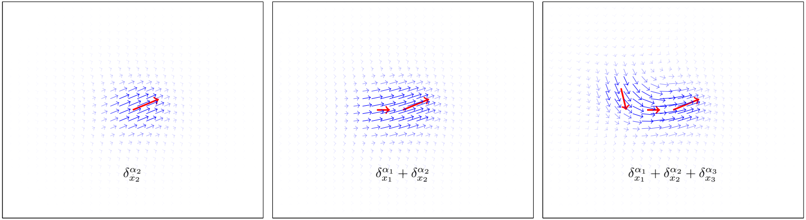


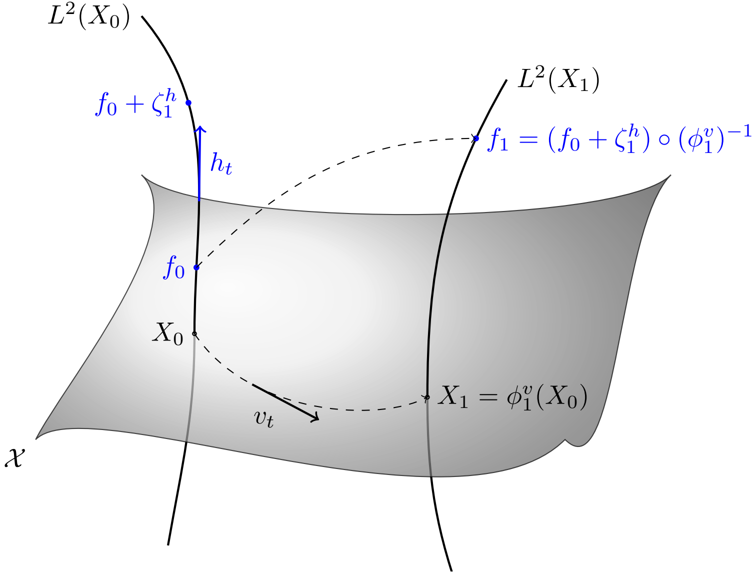
Functional deformations

- **Functional deformations :** For a given fshape $(X, f) \in \mathcal{F}$, we consider evolutions of signal functions given by

$$f_t = f + \zeta_t^h \in L^2(X)$$

for $h \in L^2([0,1], L^2(X))$ and $\zeta_t^h(x) \doteq \int_0^t h_s(x)ds$.

Fshape metamorphoses



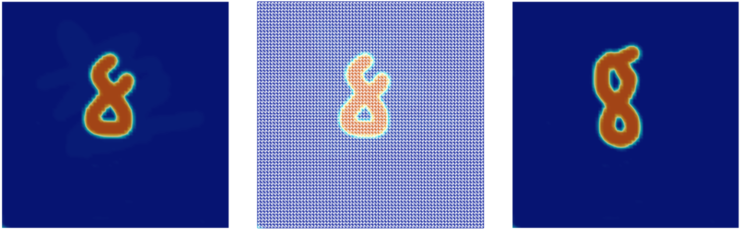
Fshape metamorphoses and metrics

- We define the energy of a transformation path (v, h) by :

$$E_X(v,h) \doteq \frac{\gamma_V}{2}\int_0^1 |v_t|_V^2 + \frac{\gamma_f}{2}\int_0^1 \int_{X_t} |h_t \circ (\phi_t^v)^{-1}|^2(x) d\mathcal{H}^d(x)$$

  where $X_t = \phi_t^v(X)$ and $\gamma_V, \gamma_f > 0$.

- Find minimum energy transformation path:



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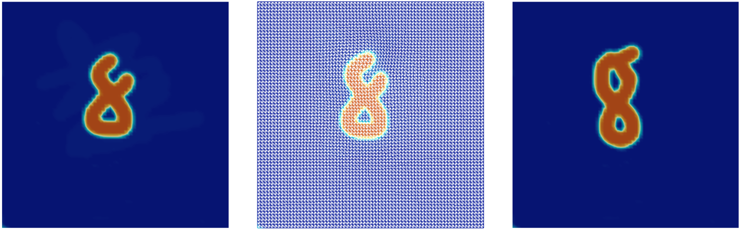

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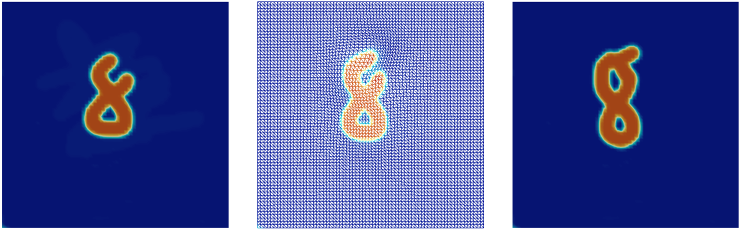

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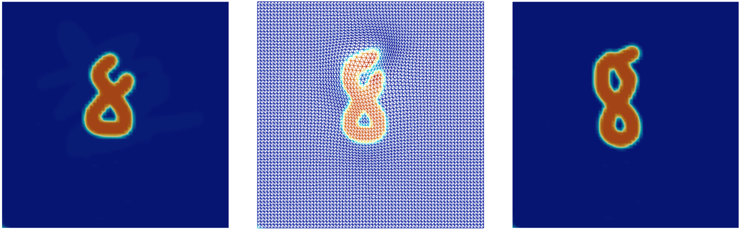

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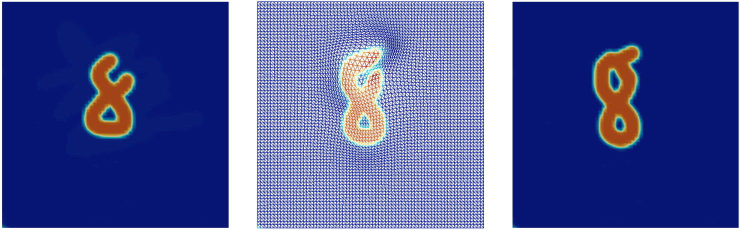

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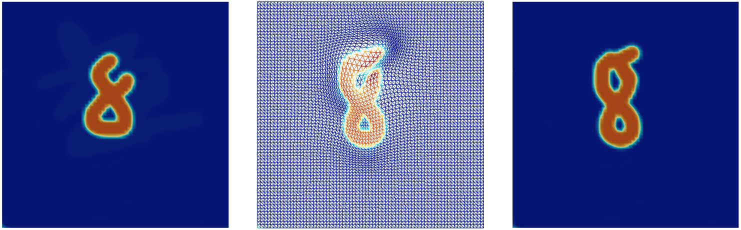

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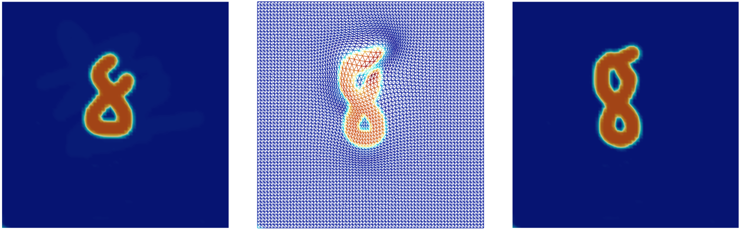

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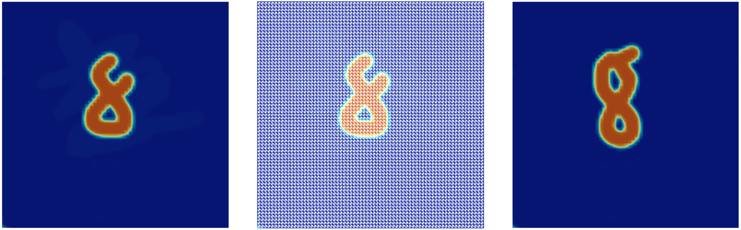

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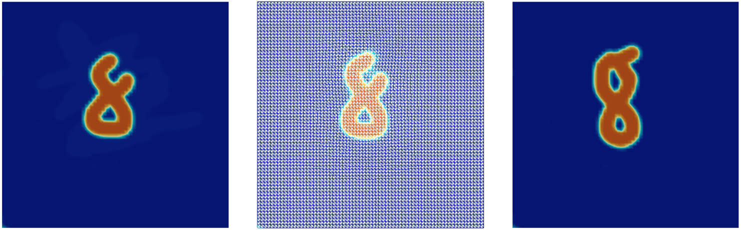

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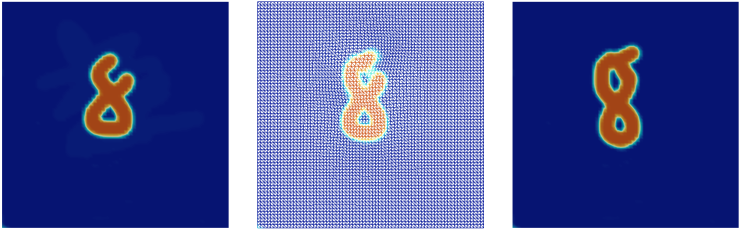

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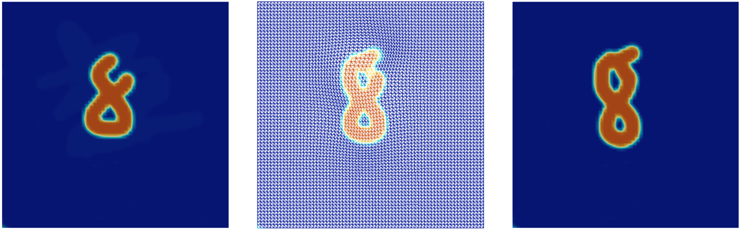
$$E_X(v,h) \doteq \frac{\gamma_V}{2} \int_0^1 |v_t|_V^2 + \frac{\gamma_f}{2} \int_0^1 \int_{X_t} |h_t \circ (\phi_t^v)^{-1}|^2(x) d\mathcal{H}^d(x)$$

where $X_t = \phi_t^v(X)$ and $\gamma_V, \gamma_f > 0$.

- Find minimum energy transformation path:



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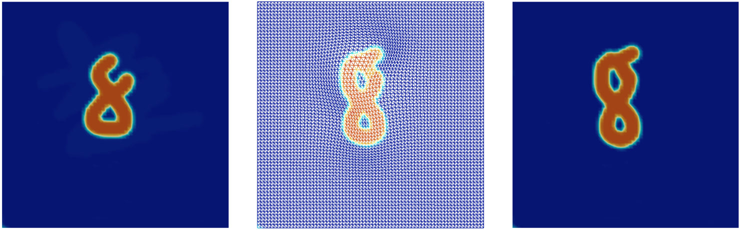

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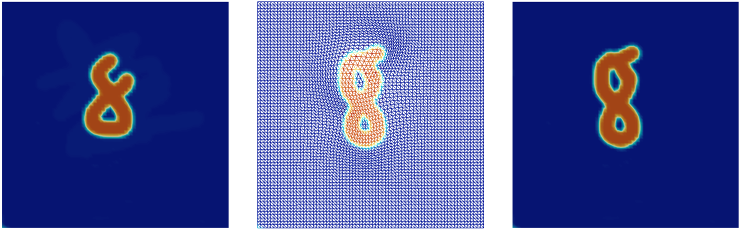

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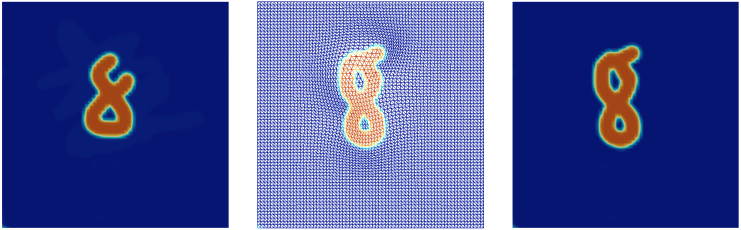

Some theoretical results

- $E_X(v,h)$ induces the following Riemannian distance on $\mathcal{F}$:

$$\begin{cases} d_{\mathcal{F}}((X,f),(X',f')) \doteq \inf \left\{ E_X(v,h)^{\frac{1}{2}} \right\} \\ \text{with } X' = \phi_1^v(X), f' = (f + \zeta_1^h) \circ (\phi_1^v)^{-1} \end{cases}$$

- Existence of geodesics between (X,f) and (X',f') (both in $\mathcal{F}$).
- Existence of Kärcher means of $(X^i, f^i)_{i=1,\dots,n}$ (all in $\mathcal{F}$).

Open question : uniqueness geodesics

Definition of fvarifold

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A *fvarifold* on $\mathbb{R}^3$ is a distribution (or measure) on the space

$$\mathbb{R}^3 \times G_2(\mathbb{R}^3) \times \mathbb{R},$$

where $G_2(\mathbb{R}^3)$ is the set of 2-dimensional subspaces of $\mathbb{R}^3$ (Grassmannian).

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- A Dirac $\delta_{(x,\vec{n},f)}$ corresponds to a singular mass located at position $x \in \mathbb{R}^3$ in the direction of the subspace $\mathrm{Vect}(\vec{n})$ carrying signal value f.
- To any non-oriented fshape (X, f) corresponds the fvarifold $\mu_{(X,f)}$ defined for all $\omega \in C_0^1(\mathbb{R}^3 \times G_2(\mathbb{R}^3) \times \mathbb{R})$:

$$\mu_{(X,f)}(\omega) = \int_X \omega(x, T_x X, f(x)) d\mathcal{H}^2(x) \approx \left(\sum_i r_i \delta_{(x_i, \vec{n}_i, f_i)} \right)(\omega)$$

Kernel based metrics

RKHS: Let W be the RKHS dense in $C_0^1(\mathbb{R}^3 \times G_2(\mathbb{R}^3) \times \mathbb{R})$ generated by a product kernel $k_e \otimes k_t \otimes k_s : (\mathbb{R}^3 \times G_2(\mathbb{R}^3) \times \mathbb{R})^2 \to \mathbb{R}$ induces a Hilbert space structure on the set of fshapes that writes:

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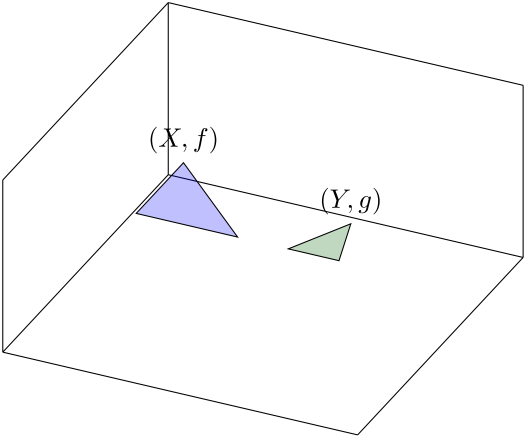
Distance: $\|\mu_{(X,f)} - \mu_{(Y,g)}\|_{W'}^2 = \left\langle \mu_{(X,f)}, \mu_{(X,f)} \right\rangle_{W'} + \left\langle \mu_{(Y,g)}, \mu_{(Y,g)} \right\rangle_{W'} - 2 \left\langle \mu_{(X,f)}, \mu_{(Y,g)} \right\rangle_{W'}.$

Discrete approximation



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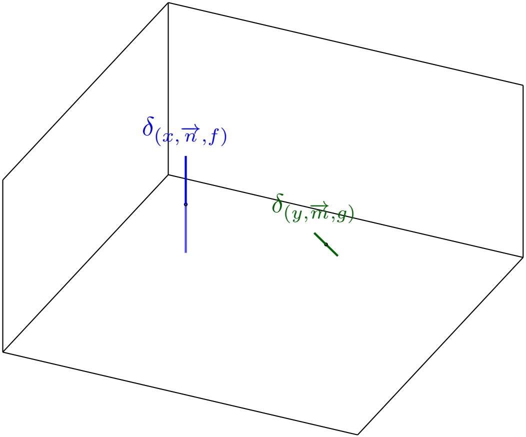

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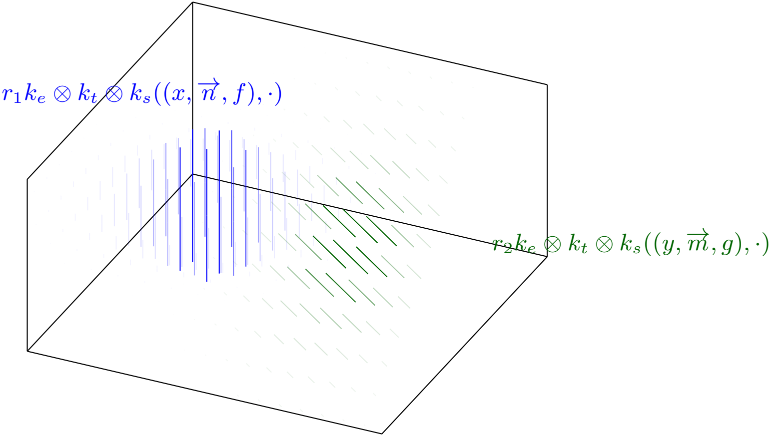

$$r_1 r_2 \langle k_e \otimes k_t \otimes k_s((x, \overrightarrow{n}, f), \cdot), k_e \otimes k_t \otimes k_s((y, \overrightarrow{m}, g), \cdot)\rangle$$
$$= r_1 r_2 k_e \otimes k_t \otimes k_s((x, \overrightarrow{n}, f), (y, \overrightarrow{m}, g))$$

Convergence of fvarifolds

lsc of varifold norm

If $f_n \to f$ a.e. on X then

$$\|\mu_{(X,f)}\|_{W'} = \lim_n \|\mu(X, f_n)\|_{W'} .$$

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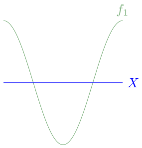
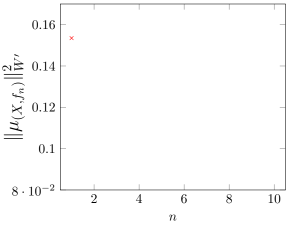


(X, f)



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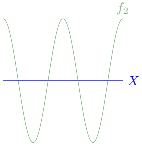

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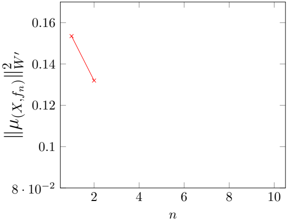

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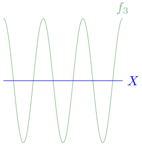

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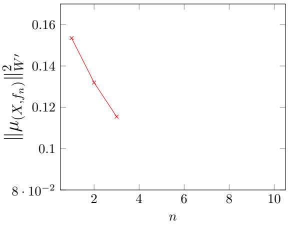

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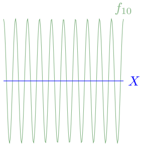

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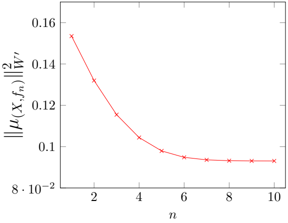

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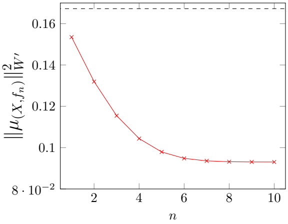

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Hyper Template

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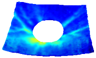

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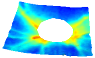

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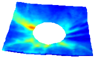

(X^3, f^3)

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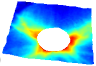

Space of solutions : Restrict the optimization to an 'orbit' of a certain **hypertemplate** fshape (X^{HT}, f^{HT}).
$\Rightarrow$ Optimization wrt $(v^0, h^0) \in L^2([0,1], V_0) \times L^2([0,1], X^{\mathrm{HT}})$.

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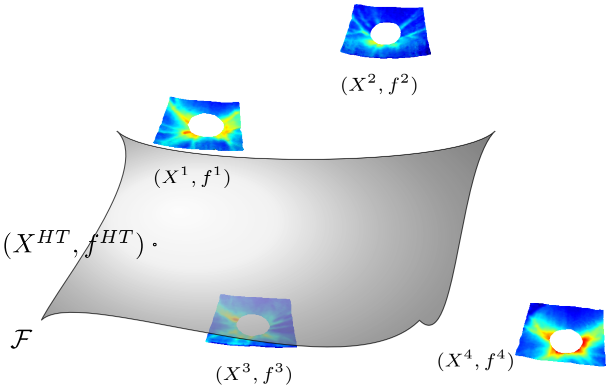

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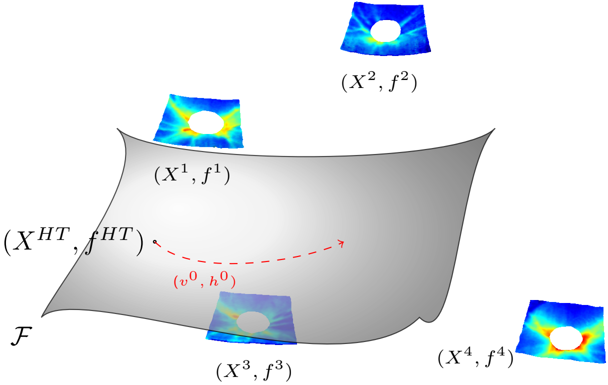

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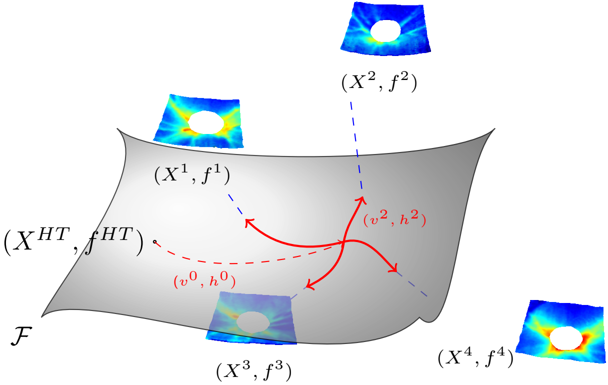

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Atlas Estimation

Variational formulation:

$$(v^0_*, h^0_*, (v^i_*, h^i_*)_i) = \underset{(v^0,h^0),(v^i,h^i)_i}{\operatorname{argmin}} E_{X^{\mathrm{HT}}}(v^0, h^0) + \sum_{i=1}^{N} \left(E_X(v^i, h^i) + \gamma_W \, g((\tilde{X}^i, \tilde{f}^i), (X^i, f^i)) \right)$$

where g is the fvarifold squared distance between subject i and $(\tilde{X}^i, \tilde{f}^i) \doteq (\phi^{v^i}, \zeta^{h^i}) \cdot (X, f)$ and $(X, f) \doteq (\phi^{v^0}, \zeta^{h^0}) \cdot (X^{\mathrm{HT}}, f^{\mathrm{HT}})$.

Theorem (Existence of solutions)

Assume that W is continuously embedded in $C^2_0(E \times G_d(E) \times \mathbb{R})$. If $\gamma_{V_0}, \gamma_V > 0$ and γ_f/γ_W and γ_ζ/γ_W are large enough, then :

- *the previous functional achieves its minimum on $v^0, \zeta^0, (v^i, \zeta^i)_{i=1,\dots,N}$*

Merci de votre attention.