Inverse Littlewood-Offord problems for quasi-norms

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(joint work with Ohad Giladi and Olivier Guédon)

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Introduction

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Our goal is to study the following object $\rho_R^K(S)$.

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To measure this, consider the Lévy concentration function of S

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Intuitively, this probability measures the amount of additive structure present between the $v_1, \ldots, v_n$.

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Studied by Lévy, Kolmogorov, Littlewood-Offord, Erdös, Esseen, Halasz, Frankl-Füredi, Tao-Vu, Rudelson-Vershynin, F.-Sodin, Nguyen-Vu, ...

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Esseen's inequality for a general random vector X says the following

$$\rho_R^{B_2^d}(X) \le C^d \left(\frac{R}{\sqrt{d}} + \sqrt{d}\right)^d \int_{B_2^d} \left|\mathbb{E}\exp\left(i\langle X, \xi\rangle\right)\right| d\xi.$$

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The following Esseen type estimate was shown in [FG11,FGG14]

$$\rho_R^K(X) \leq (\kappa(K) R)^d \int_{\mathbb{R}^d} \left| \mathbb{E} \exp(i \langle X, \xi \rangle) \right| e^{-\frac{R^2 |\xi|_2^2}{2}} d\xi.$$

where $\kappa(K) = C_K \sqrt{\frac{2}{\pi}} \left(\frac{\mu_d(K)}{\gamma_d(K)} \right)^{1/d}$, $\gamma_d(K)$ being the d-dimensional gaussian measure of K, and $\mu_d(K)$ its Lebesgue measure.

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We show that if $\rho_R^K(S)$ does not decay too fast as $n \to \infty$,

- (geometric) then many of the vectors in the set $\{v_1 \ldots, v_n\} \subseteq \mathbb{R}^d$ are “well-concentrated” around a given hyperplane.

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- (geometric) then many of the vectors in the set $\{v_1 \ldots, v_n\} \subseteq \mathbb{R}^d$ are "well-concentrated" around a given hyperplane.
- (arithmetic) then the set $\{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ can be approximated with a set which has some arithmetic structure.

First result: Forward Littlewood-Offord type for quasi-norms

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Theorem

Let $V = \{v_1, \ldots, v_n\}$ *be a multi-set of vectors in* $\mathbb{R}^d$ *satisfying that for any hyperplane* $H \subseteq \mathbb{R}^d$*, one has* $\mathrm{dist}_2(v_j, H) > R$ *for at least* k *values of* $i = 1, \ldots, n$*.*

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Then

$$\int_{\mathbb{R}^d} \left| \mathbb{E} \exp\left(i\langle S, \xi \rangle\right) \right| e^{-\frac{|\xi|_2^2}{2}} d\xi \leq \left(40 \, \frac{R+1}{R} \sqrt{\frac{d}{d+ck}} \right)^d$$

where $\mathrm{dist}_K(v, S) = \inf \left\{ \|x - s\|_K \;\middle|\; s \in S \right\}$.

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Taking contrapositives, we obtain an inverse Littlewood-Offord theorem

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Corollary (Concentration near a hyperplane)

Assume that

$$\rho_R^K(S) \geq (80\kappa(K))^d \left(\frac{d}{d+ck}\right)^{d/2}.$$

Then there exists a hyperplane $H \subset \mathbb{R}^d$ *so that*

$$\mathrm{dist}_2(v_j, H) \leq R.$$

for at least $n-k$ *values of* $i = 0, \ldots, n$.

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for at least $n-k$ values of $i = 0, \ldots, n$.

In particular, we have $\mathrm{dist}_K(v_j, H) \leq \omega_2(K)R$, where $\omega_p(K) = \inf\left\{t > 0 \;\middle|\; B_p^d \subseteq tK\right\}$ for $p > 0$.

Second result: General arithmetic progression (GAP)

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We say that a set $Q \subseteq \mathbb{R}^d$ is a general arithmetic progression (GAP), if there exist $L_1, \ldots, L_r \in \mathbb{N}$ and vectors $g_1, \ldots, g_r \in \mathbb{R}^d$ such that Q can be written in the following way.

$$Q = \left\{ \sum_{j=1}^{r} x_j g_j \;\middle|\; x_j \in \mathbb{Z},\ |x_j| \leq L_j,\ j \leq r \right\}.$$

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The number r is said to be the rank of Q, and is denoted by $\mathrm{rank}(Q)$.

Finally, the vectors $g_1, \ldots, g_r \in \mathbb{R}^d$ are said to be generators of Q.

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Theorem

Let $A, \varepsilon > 0$. Assume that

$$\rho_R^K(S) \geq n^{-A}.$$

Let $n' \in [n^\varepsilon, n]$ be a positive integer. Then there exist a GAP $Q \subseteq \mathbb{R}^d$,
a positive integer k satisfying

$$\sqrt{\frac{n'}{640\pi^2\sqrt{d\log\left(n^A\kappa(K)\right)}}} \leq k \leq \sqrt{n'},$$

and a number α
such that

Second result: Approximate arithmetic progression (cont.)

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1. Q **has small rank and cardinality:**

$$\operatorname{rank}(Q) \le C\left(d + \frac{A}{\varepsilon}\right), \quad |Q| \le C(A, d, \varepsilon)\frac{(n')^{\frac{d-\operatorname{rank}(Q)}{2}}}{\rho_R^K(S)}.$$

2. Q **approximates** V **in the** K **quasi-norm:** At least $n - n'$ elements of $v \in V$ satisfy

$$\operatorname{dist}_K(v, Q) \le C\frac{\omega_\infty(K)R}{dk}.$$

3. Q **has full dimension:** There exists $C' \le Cd\alpha$ such that

$$\{-1, 1\}^d \subseteq \frac{C'k}{R}Q.$$

4. **The generators of** Q **have bounded** K **quasi-norm:**

$$\max_{1\le j\le r} \|g_j\|_K \le C(A, d, \varepsilon)\, C_K^{k+1}\left(\frac{Dk}{R}\max_{v\in V}\|v\|_K + \omega_\infty(K)\right).$$

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- The euclidean case of the corollary is a key ingredient in the proof of Frankl and Füredi's conjecture given by Tao-Vu'12.

- It is evident that it is desirable to have good estimates on $\kappa(K)$. It could be of interest to study bodies for which $\kappa(K)$ is a constant, that is, does not depend on d.