Geostatistics for point processes
Predicting the intensity of ecological point process data

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Study of stochastic phenomena $Y = \{Y(x), x \in D \subset \mathbb{R}^d\}$, where $Y$ is a random variable measured at spatial locations $x_1, \ldots, x_n$.

Three main components:

- **Geostatistics:**
  spatial data indexed over a continuous space

- **Lattice data:** spatial data indexed over a lattice of points

- **Spatial point patterns:**
  pertaining to the location of events of interest
Motivations

Predicting the local intensity
Defining the predictor, similarly to a kriging interpolator
Solving a Fredholm equation to find the weights
⇒ approximated solutions
Illustrative results

Discussion
Motivating example

How to know the spatial distribution of a bird species at a national scale from observations made in a limited number of windows of few hectares?

The issue

How to extensively map the intensity of a point process in a large window when observation methods are available at a much smaller scale only?

⇒ The intensity of the process must be predicted from data issued out of samples spread over the window of interest.
Let $\Phi$ a point process assumed to be

- stationary and isotropic,

\[ \lambda = \frac{\mathbb{E} [\Phi(S_{obs})]}{\nu(S_{obs})} ; \quad g(r) = \frac{1}{2\pi r} \frac{\partial K^*(r)}{\partial r} \]

with $K^*(r) = \frac{1}{\lambda} \mathbb{E} [\Phi(b(0, r)) - 1|0 \in \Phi]$.

- observed in $S_{obs}$,

- driven by a stationary random field, $Z$. 
Our aim

Local intensity

We call *local* intensity of the point process $\Phi$, its intensity given the random field, $Z$: $\lambda(x|Z)$.

Window of interest:

$$S = S_{obs} \cup S_{unobs} = (\bigcup \square) \cup (\bigcup \Box)$$

$$\Phi = \{\circ, \bullet\}; \Phi_{S_{obs}} = \{\bullet\}$$

Our aim

To predict the local intensity in an unobserved window $S_{unobs}$. 
Example

Thomas process:

- $\kappa$: intensity of the Poisson process parents, $Z$,
- $\mu$: mean number of offsprings per parent,
- $\sigma$: standard deviation of Gaussian displacement.

This process is stationary with intensity $\lambda = \kappa \mu$.

The local intensity corresponds to the intensity of the inhomogeneous Poisson process of offsprings, i.e. the intensity conditional to the parent process $Z$.

More generally, we consider any process driven by a stationary random field.*
Existing solutions

- From the reconstruction of the process
  - Reconstruction method based on the 1\textsuperscript{st} and 2\textsuperscript{nd}-order characteristics of $\Phi$ (see e.g. Tscheschel & Stoyan, 2006).
  - Get the intensity by kernel smoothing.
  
  A simulation-based method $\Rightarrow$ long computation times.

- Intensity driven by a stationary random field

  Models constrained within the class of Cox processes.

- van Lieshout and Baddeley (2001).
Our alternative approach

We want to predict the local intensity $\lambda(x|Z)$

- outside the observation window,
- without precisely knowing the underlying point process $\Rightarrow$ we only consider the 1$^{st}$ and 2$^{nd}$-order characteristics,
- in a reasonable time.

We define an unbiased linear predictor

- which minimizes the error prediction variance (as in the geostatistical concept).
- whose weights depend on the structure of the point process.
Our predictor

**Proposition**

The predictor $\hat{\lambda}(x_o | Z) = \sum_{x \in \Phi \cap S_{obs}} w(x)$ is the BLUP of $\lambda(x_o | Z)$.

The weights, $w(x)$, are solution of the Fredholm equation of the 2$^d$ kind:

$$w(x) + \lambda \int_{S_{obs}} w(y) (g(x - y) - 1) \, dy - \frac{1}{\nu(S_{obs})} \left[ 1 + \lambda \int_{S_{obs}^2} w(y) (g(x - y) - 1) \, dx \, dy \right]$$

$$= \lambda (g(x_o - x) - 1) - \frac{\lambda}{\nu(S_{obs})} \int_{S_{obs}} (g(x_o - x) - 1) \, dx$$

and satisfy $\int_{S_{obs}} w(x) \, dx = 1$. 
Solving the Fredholm equation

Any existing solution already considered in the literature can be used!

Our aim is to map the local intensity in a given window ⇒ access to fast solutions.

Several approximations can be used to solve the Fredholm equation.

The weights \( w(x) \) can be defined as

- step functions \( \rightsquigarrow \) direct solution,
- linear combination of known basis functions, e.g. splines \( \rightsquigarrow \) continuous approximation.
- ...

Here, we illustrate the ones with the less heavy calculations and implementation.
Step functions

Let consider the following partition of $S_{obs}$: $S_{obs} = \bigcup_{j=1}^{n} B_{j}$, with

- $B$: elementary square centered at 0,
- $B_{j} = B \oplus c_{j}$: elementary square centered at $c_{j}$,
- $B_{k} \cap B_{j} = \emptyset$,
- $n$: number of grid cell centers lying in $S_{obs}$.

For $w(x) = \sum_{j=1}^{n} w_{j} \frac{1_{\{x \in B_{j}\}}}{\nu(B)}$, we get

$$\hat{\lambda}(x_{o}|Z) = \sum_{j=1}^{n} w_{j} \frac{\Phi(B_{j})}{\nu(B)},$$

with $w = (w_{1}, \ldots, w_{n}) = C^{-1}C_{o} + \frac{1-T^{T}C^{-1}C_{o}}{1+C^{-1}1}C^{-1}1$, where

- $C = \lambda \nu(B)I + \lambda^{2} \nu^{2}(B) (G - 1)$: covariance matrix with $G = \{g_{ij}\}_{i,j=1,\ldots,n}$, $g_{ij} = \frac{1}{\nu^{2}(B)} \int_{B \times B} g(c_{i} - c_{j} + u - v) \, du \, dv$, and $I$ the $n \times n$-identity matrix.

- $C_{o} = \lambda \nu(B)I_{x_{o}} + \lambda^{2} \nu^{2}(B) (G_{o} - 1)$: covariance vector with $I_{x_{o}}$ the $n$-vector with zero values and one term equals to one where $x_{o} = c_{i}$, and $G_{o} = \{g_{io}\}_{i=1,\ldots,n}$. 

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Step functions: variance of the predictor

We consider the Neuman series to invert the covariance matrix, $C = \lambda \nu(B)\mathbb{I} + \lambda^2 \nu^2(B)(G - 1)$, when $\lambda \nu(B) \to 0$:

$$C^{-1} = \frac{1}{\lambda \nu(B)} [\mathbb{I} + \lambda \nu(B)J_\lambda] ,$$

where a generic element of the matrix $J_\lambda$ is given by

$$J_\lambda[i,j] = \sum_{k=1}^{\infty} (-1)^k \lambda^{k-1} (g(x_i, x_{l_1}) - 1) \left(g(x_{l_{k-1}}, x_j) - 1\right)$$

$$\times \int_{S_{obs}^{k-1}} \prod_{m=1}^{k-2} (g(x_{l_m}, x_{l_{m+1}}) - 1) \; dx_{l_1} \cdots dx_{l_{k-1}} .$$

This leads to

$$\text{Var} \left( \hat{\lambda}(x_o | Z) \right) = \lambda^3 \nu^2(B)(G_o - 1)^T (G_o - 1) + \lambda^4 \nu^3(B)(G_o - 1)^T J_\lambda(G_o - 1)$$

$$+ \frac{1 - \left[ \lambda \nu(B) \mathbf{1}^T (G_o - 1) + \lambda^2 \nu^2(B) \mathbf{1}^T J_\lambda(G_o - 1) \right]^2}{\frac{\nu(S_{obs})}{\lambda} + \nu^2(B) \mathbf{1}^T J_\lambda \mathbf{1}} .$$
Step functions: illustrative results about prediction

Simulated Thomas process
\( \kappa = 10, \mu = 50, \sigma = 0.05 \)

Theoretical local intensity

\[ S = S_{obs} \cup S_{unobs} = (\bigcup \square) \cup (\bigcup \square) \]

Prediction within \( S_{unobs} \)

\{•\}: \( \Phi \cap S_{obs} \); \{○\}: \( \Phi \cap S_{unobs} \)
Let consider that the weights of $\hat{\lambda}(x_0|Z) = \sum_{x \in \Phi \cap S_{obs}} w(x)$ are defined as a degree $d$ spline curve:

$$w(x) = \sum_{i=1}^{k} h_{i,d}(x),$$

where $h_{i,d}$ denotes the $i$th $B$-spline of order $d$.

A simplistic toy example in $\mathbb{R}$:

- $S_{obs} = [0, L) \subset [0, L'] = S$
- Linear spline defined from equally-spaced knots $x_i$:

$$w(x) = \begin{cases} 
  a_0 + b_0 x, & x \in \Delta_0 = [x_0, x_1) = [0, \frac{L}{k}), \\
  a_1 + b_1 x, & x \in \Delta_1 = [x_1, x_2) = [\frac{L}{k}, \frac{2L}{k}), \\
  \vdots & \\
  a_{k-1} + b_{k-1} x, & x \in \Delta_{k-1} = [x_{k-1}, x_k) = [\frac{(k-1)L}{k}, L), \\
  (a_i + b_i(x - x_i)) 1_{\{x \in \Delta_i\}} & 
\end{cases}$$
From the continuity property and the constraint $\int_{S_{\text{obs}}} w(x) \, dx = 1$:

$$w(x) = \frac{1}{L} - \sum_{j=0}^{k-1} b_j P_j(x),$$

with $P_j(x) = \sum_{i=0}^{k-1} \left( \frac{1/2-k+j}{k^2} - 1 \{j<i\} - (x - \frac{il}{k}) 1 \{i = j\} \right) 1 \{x \in \Delta_i\}$.

The Fredholm equation becomes

$$\sum_{j=0}^{k-1} b_j \left[ P_j(x) + \lambda \int_L P_j(y)(g(x - y) - 1) \, dy - \frac{1}{L} \int_{L^2} P_j(y)(g(x - y) - 1) \, dx \, dy \right]$$

$$= \frac{\lambda}{L} \int_L (g(x - y) - 1) \, dy - \frac{1}{L^2} \int_{L^2} (g(x - y) - 1) \, dx \, dy - \lambda (g(x_0 - x) - 1)$$

$$+ \frac{1}{L} \int_L (g(x_0 - x) - 1) \, dx$$

i.e. of the form $\sum_{j=0}^{k-1} b_j A_j(x) = Q(x)$,

Then, $(b_0, \ldots, b_{k-1}) = b$ is obtained from $m$ control points and satisfy

$$b = (X^T X)^{-1} X^T Y,$$

with $X = (A_j(x_l))_{l=1}^{m}$ and $Y = (Q(x_l))_{l=1}^{m}$. 

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Spline basis: illustrative results

Thomas process in 1D ($\kappa = 0.5, \mu = 25, \sigma = 0.25$)

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Theoretical local intensity on $S_{obs}$; Predicted values; Intensity of $\Phi$

$\{\bullet\} = \Phi_{S_{obs}}$; $\{\circ\} = \Phi_{s_{unobs}}$

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In practice: $g$ must be estimated

**Simulated Thomas process**
$\kappa = 10, \mu = 50, \sigma = 0.05$

**Prediction within $S_{unobs}$**
(with the theoretical pcf)

$R^2 = 0.85$

**Estimated pcf**

(with the estimated pcf)

$R^2 = 0.8$

$\{\bullet\}: \Phi \cap S_{obs}$; \ $\{\circ\}: \Phi \cap S_{unobs}$

$R^2$ in linear regression of predicted and theoretical values
(with the theoretical pcf)

- 24 x 24
- 48 x 48
- 96 x 96

Grid size
Montagu’s Harriers’ nest locations

Data collection
LTER Zone Atelier ‘Plaine & Val de Sèvre’

Estimated pcf

Prediction

\{ \bullet \} = \Phi S_{obs}; \{ \circ \} = \Phi S_{unobs}
Work in progress

- Take into account some covariates in the prediction.
- Get results with splines on the plane.
- Use finite elements method to solve the Fredholm equation.
- Determine the properties of the related predictor.
- Extend the approach to the spatio-temporal setting.


E. Gabriel et al. (2016) Adapted kriging to predict the intensity of partially observed point process data. To appear in *Spatial statistics*.

