Désordre et phénomènes critiques : les modèles d'accrochage

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General issue

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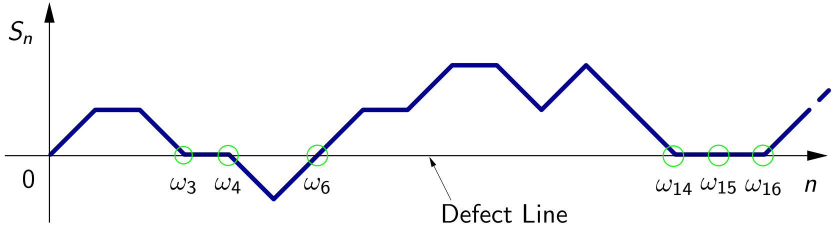
- for which much has been done when disorder is introduced
- interesting from a modeling viewpoint

"The Example": $(1+d)$–directed walk models

Symmetric Random Walk $\{S_n\}_n$ with increments in $\{-1, 0, +1\}$ $(d = 1)$



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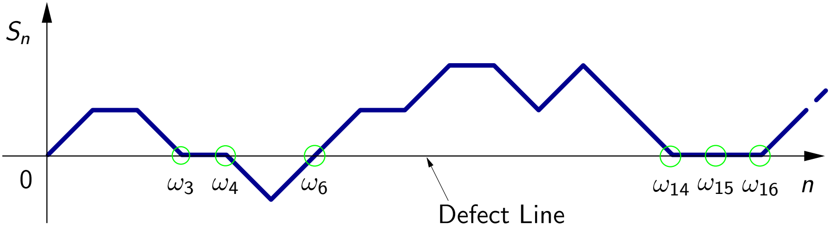

Model ($\beta \geq 0$, $h \in \mathbb{R}$):

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z_{N,\omega}} \exp\left(\sum_{n=1}^{N} (\beta\omega_n + h)\,\delta_n\right)$$

with $\delta_n = \mathbf{1}_{S_n=0}$. The disorder ω is a IID sequence $\mathcal{N}(0,1)$ of law $\mathbb{P}$.

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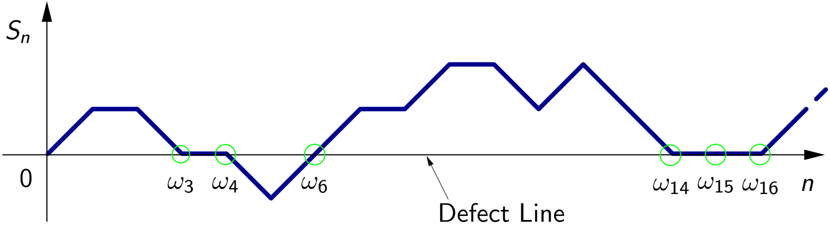

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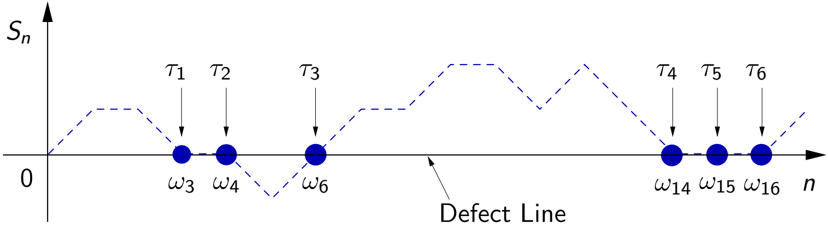

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with $\delta_n = \mathbf{1}_{n \in \tau}$.

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Basic building block: a Discrete Renewal Process

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1. $\tau = \{\tau_0, \tau_1, \tau_2, \dots\}$ discrete renewal sequence (that is, $\tau_0 = 0$ and $\{\tau_j - \tau_{j-1}\}_{j \in \mathbb{N}}$ is IID), of law $\mathbf{P}$, s. t.

$$K(n) = \mathbf{P}(\tau_1 = n) \sim C_K / n^{1+\alpha}, \quad (C_K > 0),$$

and

$$\sum_{n \in \mathbb{N}} K(n) \le 1.$$

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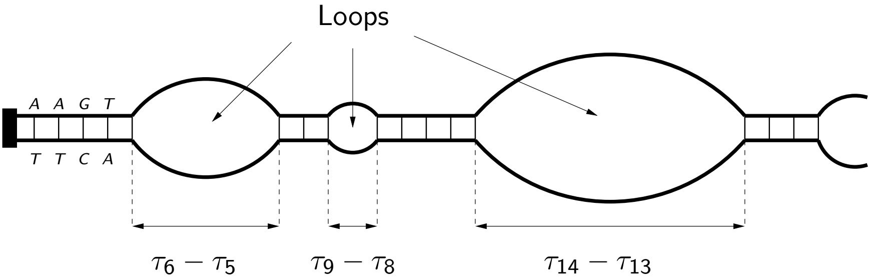
2. If $\sum_n K(n) < 1 \Longrightarrow$ renewal on $\mathbb{N} \cup \{\infty\}$, with $K(\infty) = 1 - \sum_n K(n)$ *(terminating renewal)*, otherwise the renewal is *persistent*.

Obs. : $\alpha = 1/2$ for both $d = 1$ and 3, but $\sum_n K(n) < 1$ if $d = 3$

The Poland-Scheraga model (DNA denaturation)



- The two thick lines are the DNA strands. They may be paired, gaining thus energetic contributions that depend on whether the base pair is A-T or G-C.
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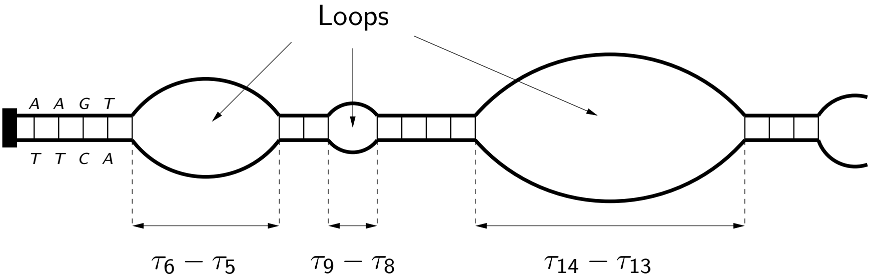

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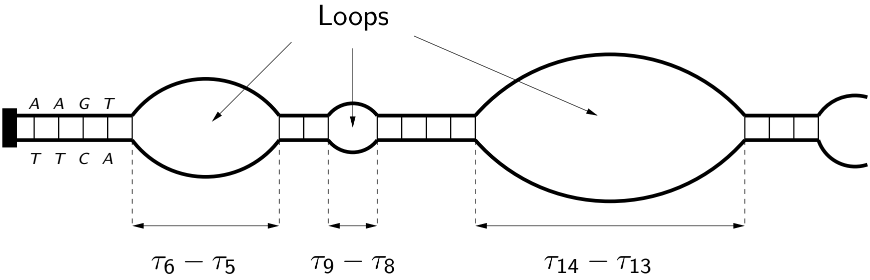

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Annealed, pure and homogeneous models

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$$\mathbb{E}Z_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}((\beta^2/2) + h)\delta_n\right)\right]$$

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So:

- The annealed (or *pure*) model is just a homogeneous model with pinning potential $h + \beta^2/2$;
- Homogeneous pinning models are *exactly solvable*

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- $\mathrm{F}(\cdot,\cdot)$ is convex, $\mathrm{F}(\beta,\cdot)$ and $\mathrm{F}(\cdot,h)$ are non-decreasing
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$$\mathrm{F}(0, h) \leq \mathrm{F}(\beta, h) \leq \mathrm{F}(0, h + \beta^2/2)$$

- $\mathrm{F}(\beta, h) \geq 0$, so either $= 0$ or > 0

Localization and Delocalization

Proof of $\textsc{F}(\beta, h) \geq 0$:

$$\begin{aligned}
\textsc{F}(\beta, h) &= \lim_N \frac{1}{N} \mathbb{E} \log \mathbf{E}\left[\exp\left(\sum_{n=1}^N (\beta\omega_n + h)\delta_n\right)\right] \\
&\geq \liminf_N \frac{1}{N} \mathbb{E} \log \mathbf{E}\left[\exp\left(\sum_{n=1}^N (\beta\omega_n + h)\delta_n\right);\, \tau_1 > N\right] \\
&= \lim_N \frac{1}{N} \log \mathbf{P}\left(\tau_1 > N\right) \,=\, 0. \qquad \square
\end{aligned}$$

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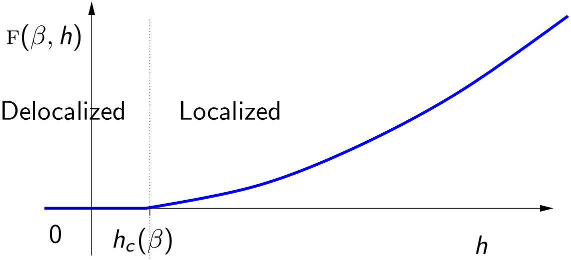

Main questions

Plenty of questions, but above all:

- Can one compute or estimate $h_c(\beta)$?
- Critical behavior? $\textsc{f}(\beta, h) \overset{h \searrow h_c(\beta)}{\sim} const.(h - h_c(\beta))^{\nu_q}$
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The pure (homogeneous, annealed) model is solvable

Semi-explicit formula for $\mathrm{F}(0, h)$ from which

- $h_c(0) = -\log \sum_n K(n) (\geq 0)$
- $$\mathrm{F}(0, h) \stackrel{h \searrow h_c(0)}{\sim} const.(h - h_c(0))^{\nu_a}, \quad \nu_a = \max(1/\alpha, 1)$$

...M. Fisher '84. But: Erdos, Pollard, Feller, Garsia, Lamperti... (40's...)

General principles to deal with disorder(?)

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HC for pinning models [Forgacs et al. (1986), Derrida et al. (1992)]:

- $h_c(\beta) = h_c^a(\beta)$ and $\nu_q = \nu_a$ for $\beta \leq \beta_0$ if $\alpha < 1/2$
- $h_c(\beta) > h_c^a(\beta)$ and (probably) $\nu_q \neq \nu_a$ for $\beta > 0$ and $\alpha > 1/2$.
- marginal case (controversial issue for physicists)

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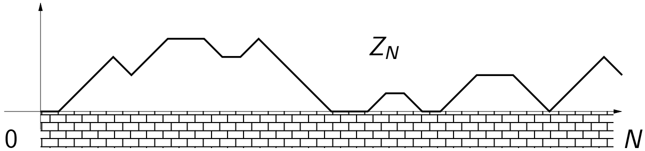
[Alexander, Berger, Biskup, Bolthausen, Caravenna, Derrida, G., den Hollander, Lacoin, Opoku, Pétrélis, Poisat, Sohier, Sun, Toninelli, Torri, Zygouras: 2004-ongoing]

Recent developments: higher dimensional

- Pinning of $d+1$ dimensional interfaces (free field, SOS type models) [Coquille, Milos], [G., Lacoin]



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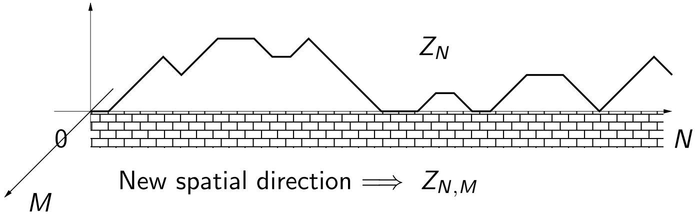

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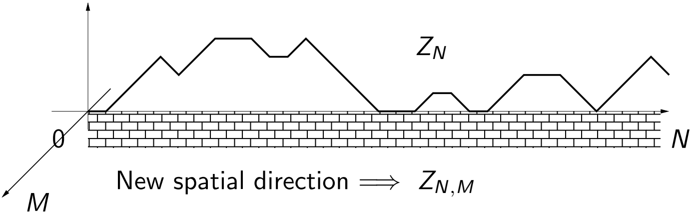


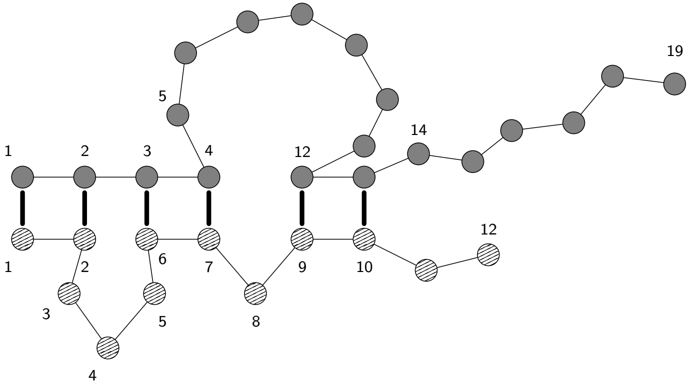
- Pinning of multi-dimensional renewal: (multi $\longrightarrow$ two)

$$\frac{\mathrm{d}\mathbf{P}_{N,M,\omega}}{\mathrm{d}\mathbf{P}}(\tau) = \frac{1}{Z_{N,M,\omega}} \exp\left(\sum_{n=1}^{N} \sum_{m=1}^{M} (\beta \omega_{n,m} + h)\, \delta_{n,m} \right)$$

with $\delta_{n,m} = \mathbf{1}_{(n,m)\in\tau}$, with τ is a two dimensional renewal

DNA denaturation and two-dimensional renewal

Generalized Poland Scheraga model [Garel, Orland 2004]



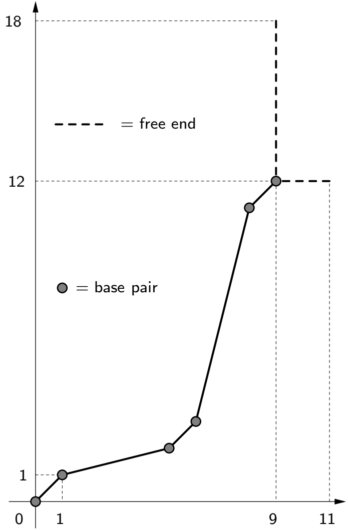
- Unequal strand length (12 and 19)
- The configuration is determined by the six base pairs $(1, 1)$, $(2, 2)$, $(6, 3)$, $(7, 4)$, $(9, 12)$ and $(10, 13)$.

DNA denaturation and two-dimensional renewal

Same trajectory,

two-dimensional renewal representation.

A d-dimensional renewal is a d-dim. walk with (componentwise) positive increments like in the one dimensional case ($d = 1$)



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The free energy density is ($\gamma > 0$)

$$\mathrm{F}_\gamma(\beta, h) := \lim_{\substack{N,M\to\infty \\ M/N\sim\gamma}} \frac{1}{N} \mathbb{E} \log Z_{N,M,\omega}$$

and the localization transition is again between $\mathrm{F}_\gamma(\beta, h) = 0$ and $\mathrm{F}_\gamma(\beta, h) > 0$.

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Again (a priori somewhat surprising), homogeneous ($\beta = 0$) exactly solvable, but richer behavior!
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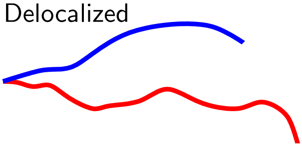
- Other non analyticicities (i.e., phase transitions) when $\mathrm{F}(0, h) > 0$
- Harris criterion program for IID disorder: this time disorder is irrelevant for $\alpha < 1$, i.e. $h_c(\beta) = h_c(0) + \beta^2/2$ for $\beta < \beta_0$ and $\nu_q = \nu_a$, and relevant for $\alpha > 1$, i.e. $h_c(\beta) < h_c(0) + \beta^2/2$ for $\beta > 0$

Two-dimensional renewal pinning: paths and transitions



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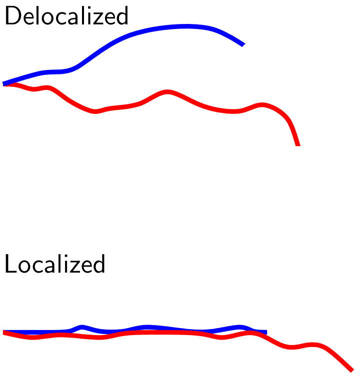

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Sum-up

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- Another major open issue: making the Harris criterion program rigorous for other classes of statistical mechanics systems