Parametric estimation in movement ecology
Focus on models based on stochastic differential equations

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Context: Movement Ecology

Main objective

- Observing movements to answer ecological/management questions.

Observing moving individuals

- Getting precise location of an individual at a precise time.

	Longitude	Latitude	Date	
X_1	-1.234	49.156	19/05/2010 04:13:12	t_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_N	-2.314	48.236	19/05/2010 23:23:41	t_N

Context: Home range analysis

The Home Range concept (Burt 1943)

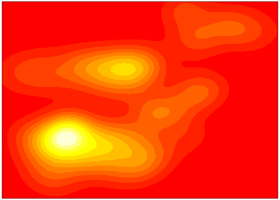
That area traversed by an individual in its normal activities of food gathering, mating, and caring for young.

From points to maps

- From GPS tracking;
- → Mapping the use of space;
- Quantification of individual's presence.

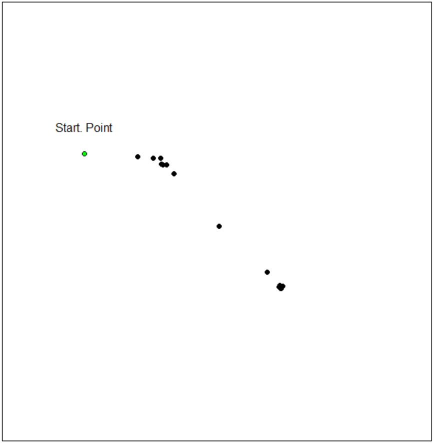
Mechanistic approach of home range

- Linking displacements to home range;
- Defining movement models depending on space;



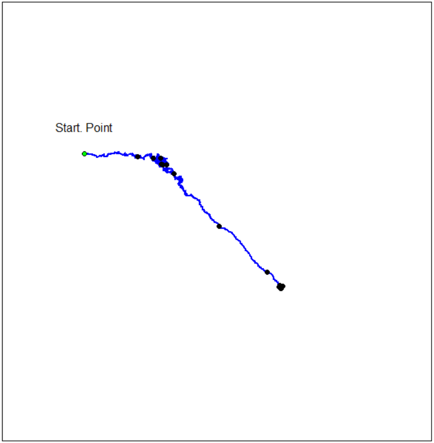
Movement models based on potential functions

Observed trajectory: Observations at discrete time.



Movement models based on potential functions

A trajectory: *A continuous process* observed at discrete times.



From the trajectory to the map

- **Assumption:**
  The trajectory mainly follows the gradient of an unknown map.

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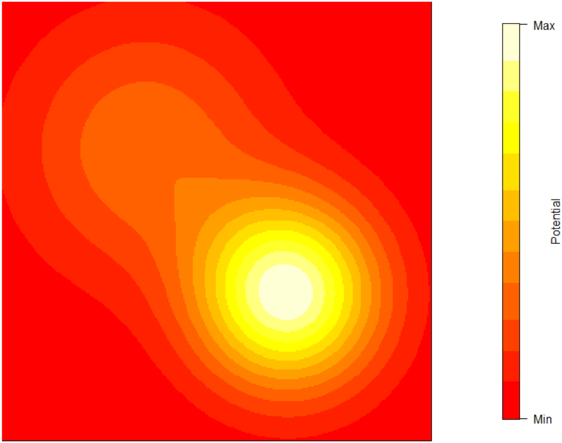

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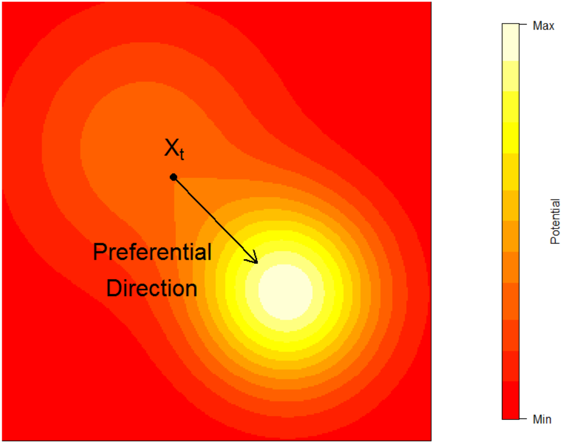

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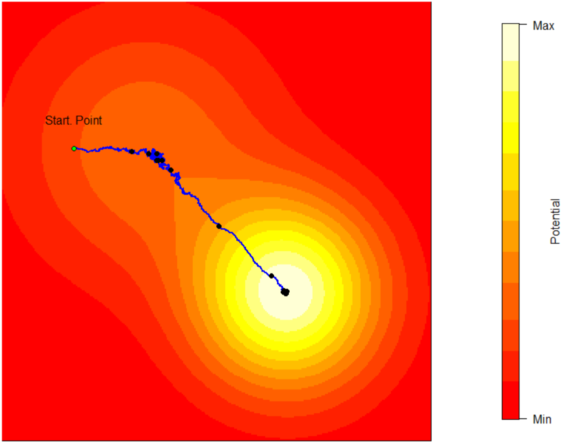

Goal: Estimate the map from observed trajectories.

Formal continuous time and space model

Model

- The 2D position process $(\mathcal{X}_t)_{t\geq 0}$ of an individual, starting at X_0 is the solution of the 2D-Stochastic Differential Equation (SDE)

$$d\mathcal{X}_t = \nabla P(\mathcal{X}_t, \theta)dt + \gamma dW_t$$

- Deterministic part (Brillinger, 2010)
  - ∇ is the gradient operator;
  - $P(\cdot) : \mathbb{R}^2 \mapsto \mathbb{R}$ is the potential map, independent of time;
  - Depends on unknown parameters θ.
- Stochastic part
  - γ is a diffusion parameter;
  - W is the 2-D standard Brownian motion.

Observations

- The continuous process $(\mathcal{X}_t)_{t\geq 0}$ is observed at discrete times $t_0, \ldots, t_n = T, \ X_{obs} = X_0, \ldots, X_T$.

MLE for discretely observed SDE

Observations

- The continuous process $(\mathcal{X}_t)_{t\geq 0}$ is observed at discrete times $t_0, \dots, t_n,\ X_{obs} = X_0, \dots, X_n$;
- $\mathcal{X}_0$ is supposed deterministic ($= X_0$)

By Markov property of the solution to the SDE, the loglikelihood is:

$$l(\theta|X_{obs}) = \sum_{i=0}^{n-1} \log p_\theta(X_{i+1}|X_i, \Delta_i)$$

where

- $\Delta_i := t_{i+1} - t_i$
- $p_\theta(x|X_i, \Delta_i)$ is the transition density, i.e., the p.d.f. of $\mathcal{X}_{i+1}|\mathcal{X}_i = X_i$;

Problem

- Except in rare cases (e.g., Constant or linear drift), p_θ is unknown;
- $\Rightarrow$ Require approximation of the MLE.

State of the art in movement ecology

A more and more used framework

(Blackwell, 1997; Blackwell et al., 2015; Brillinger et al., 2001, 2002, 2011; Harris and Blackwell, 2013; Preisler et al., 2004, 2013)

Inference methods for the MLE used in Ecology

- Explicit, if possible (Brownian motion, Ornstein Ulhenbeck process);
- Euler approximation;
- No use of other existing methods;
- GPS sampling might not be well suited for Euler method.

Question and objectives

- How robust to low frequency sampling is the Euler method on potential based models?
- Are other existing methods more robust to low frequency sampling?

Focus on four methods

- Euler Maruyama method;
- Kessler method;
- Local Linearization (Ozaki) method;
- MCEM approach using Exact algorithm.

The Euler-Maruyama method

Target SDE

$$d\mathcal{X}_t = b_\theta(\mathcal{X}_t)dt + \gamma dW_t \tag{1}$$

Approximation

$\forall i = 0, \dots, n-1$ eq. (1) is replaced by

$$\tilde{X}_i = X_i \quad \text{and} \quad \mathrm{d}\tilde{\mathcal{X}}_t = b_\theta(X_i)\mathrm{d}t + \gamma \mathrm{d}W_t \quad , \; t_i \leq t < t_{i+1}. \tag{2}$$

The transition density of the solution of (2) is known:

$$\tilde{p}_{\theta,i}\left(x, |\tilde{X}_i, \Delta_i\right) \text{ p.d.f. } \mathcal{N}(\mu_i, \Sigma_i)$$

$$\mu_i = X_i + b_\theta(X_i) \times \Delta_i, \quad \Sigma_i = \text{diag}(\gamma^2 \Delta_i)$$

Therefore the estimate is given by

$$\hat{\theta}_{Euler} = \text{argmax}_\theta \sum_{i=0}^{n-1} \log \tilde{p}_{\theta,i}(X_{i+1}|X_i, \Delta_i)$$

The Kessler method, (Kessler, 1997)

Approximation

$\forall i = 0, \dots, n-1$ the target SDE is replaced , for $t_i \leq t < t_{i+1}$

$$\tilde{X}_i = X_i, \quad \mathrm{d}\tilde{\mathcal{X}}_t = \frac{\mathbb{E}(\mathcal{X}_{i+1}|\mathcal{X}_i = X_i) - X_i}{\Delta_i}\mathrm{d}t + \mathbb{V}(\mathcal{X}_{i+1}|\mathcal{X}_i = X_i)^{\frac{1}{2}}\mathrm{d}W_t \quad . \tag{3}$$

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$$\mu_i = \mathbb{E}(\mathcal{X}_{i+1}|\mathcal{X}_i = X_i), \quad \Sigma_i = \mathbb{V}(\mathcal{X}_{i+1}|\mathcal{X}_i = X_i)$$

Florens-zmirou (1989) gives an expansion of these two moments,

$$\Rightarrow \hat{\theta}_{kessler} = \mathrm{argmax}_\theta \sum_{i=0}^{n-1} \log \tilde{p}_{\theta,i}(X_{i+1}|X_i, \Delta_i)$$

The Local linearization method, (Ozaki, 1992)

Approximation

$\forall i = 0, \ldots, n-1$ the target SDE is replaced , for $t_i \leq t < t_{i+1}$

$$\tilde{X}_i = X_i, \quad \mathrm{d}\tilde{X}_t = J_{i,\theta}\left[\tilde{X}_t - X_i + (J_{i,\theta_P})^{-1} b_\theta(X_i)\right]\mathrm{d}t + \gamma \mathrm{d}W_t \quad . \tag{4}$$

where $J_{i,\theta} = \frac{\delta b}{\delta x}(X_i)$

The transition density of the solution of (4) is known:

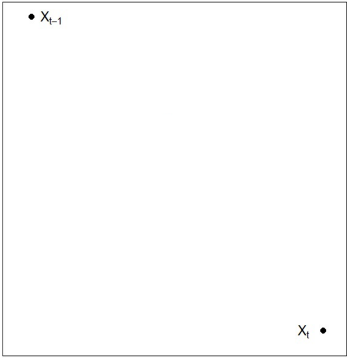
$$\tilde{p}_{\theta,i}\left(x, |\tilde{X}_i, \Delta_i\right) \text{ p.d.f. } \mathcal{N}(\mu_i, \Sigma_i)$$

$$\mu_i = X_i + (\exp(J_{i,\theta}) - I_2)(J_{i,\theta})^{-1} b_\theta(X_i),$$

$$\mathrm{vec}(\Sigma_i) = (J_{i,\theta} \oplus J_{i,\theta})^{-1}\left(e^{(J_{i,\theta} \oplus J_{i,\theta})\Delta_i} - I_2\right)\mathrm{vec}(\gamma^2 I_2)$$

$$\Rightarrow \hat{\theta}_{Ozaki} = \mathrm{argmax}_\theta \sum_{i=0}^{n-1} \log \tilde{p}_{\theta,i}(X_{i+1}|X_i, \Delta_i)$$

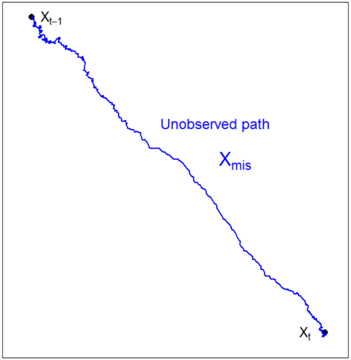
The Exact Algorithm approach, (Beskos et al., 2006)



Problem

- $p(X_t|X_{t-1}, \theta)$ is unknown;
- $\Rightarrow l(\theta|X_{t-1}, X_t)$ is unknown.

The Exact Algorithm approach, (Beskos et al., 2006)



Alternative

- Suppose X_{mis} is known;
- The complete log likelihood $l(\theta|X_{i-1}, X_i, X_{mis})$ can be written.
- $\Rightarrow \mathbb{E}_{\mathcal{X}_{mis}}(l(\theta|X_{i-1}, X_i, X_{mis}))$ can be computed.

Complete Log Likelihood (Girsanov + Ito Lemma)

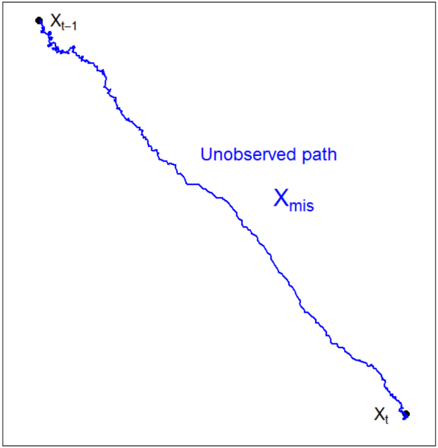
$$l(\theta|X_{i-1}, X_i, {}_{mis}) = P(X_i, \theta) - P(X_{i-1}, \theta) - \frac{1}{2}\int_{t_i-1}^{t_i} c(X_s, \theta)ds$$

where $c(X_s, \theta) := \| \nabla P(X_s, \theta) \|^2 + \triangle P(X_s, \theta)$

The Exact Algorithm approach, (Beskos et al., 2006)

Solution: EM algorithm

Maximising iteratively $\mathbb{E}_{\mathcal{X}_{mis}}(l(\theta|X_{i-1}, X_i, \mathcal{X}_{mis}))$; leads to the MLE.



Monte Carlo approach

- $\mathbb{E}_{\mathcal{X}_{mis}}(\cdot)$ has to be approximated;
- Need to simulate $\mathcal{X}_{mis}$ **conditionally to** (X_{i-1}, X_i);

The Exact Algorithm approach, (Beskos et al., 2006)

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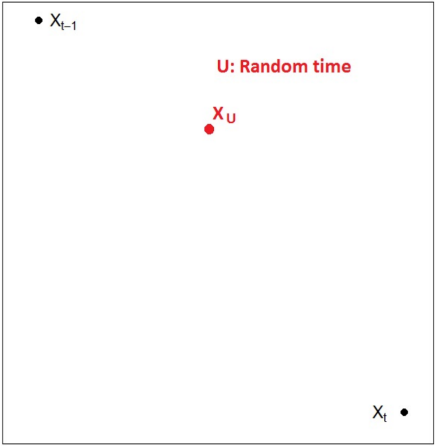


Monte Carlo approach

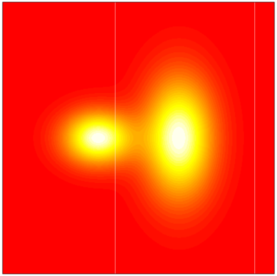
- $\mathbb{E}_{\mathcal{X}_{mis}}(\cdot)$ has to be approximated;
- Need to simulate $\mathcal{X}_{mis}$ **conditionally to** (X_{i-1}, X_i);
- $\mathbb{E}_{\mathcal{X}_{mis}}(\cdot) = \mathbb{E}_{\mathcal{X}_U}(\cdot), U \sim \mathcal{U}[t_{i-1}, t_i]$
- Exact simulation algorithm (Beskos et al. 2006);
- E step is performed;

Resulting $\hat{\theta}_{EA}$ is an unbiased estimator of the MLE.

Parametric form for the potential map

A mixture of Gaussian forms (inspired by Preisler et al., 2013)

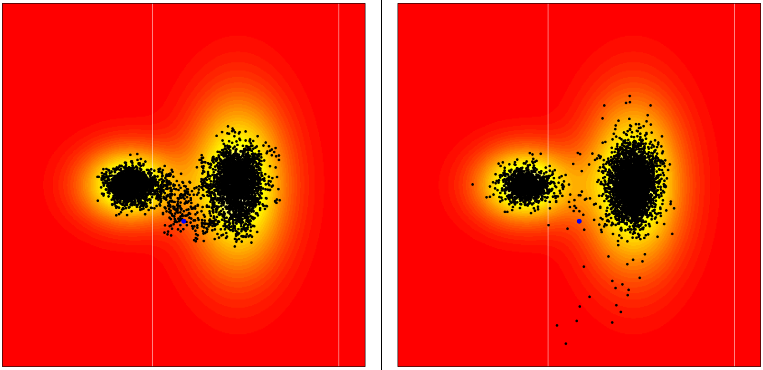
- $P(X, \theta) = \sum_{k=1}^{K} \pi_k \exp\left(-\frac{1}{2}(X - \mu_k)^T C_k (X - \mu_k)\right)$
- K is the (**given**) number of components (attractive zones);
- μ_k is the location of the k-th attractive zone;
- C_k is a covariance matrix, the shape of k-th attractive zone;
- π_k is the (positive) weight of the k-th attractive zone.



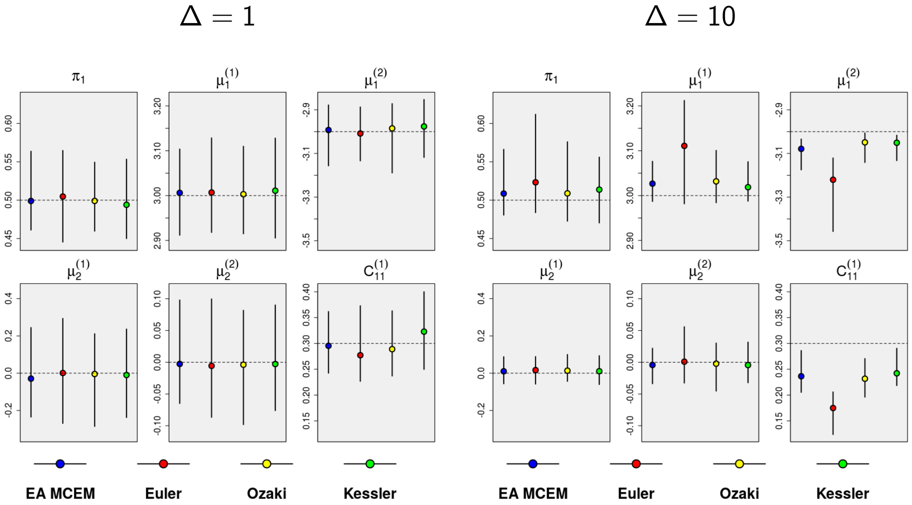
Simulation

- $K = 2$ attractive zones (12 parameters to estimate);
- 10 independant realization of the SDE (exactly) simulated;
- 500 discrete observations per trajectory of the SDE (5000 pts total);
- Two samplings considered, $\Delta = 1$ and $\Delta = 10$;
- Experience repeated 30 times.

$\Delta = 1$ $\Delta = 10$

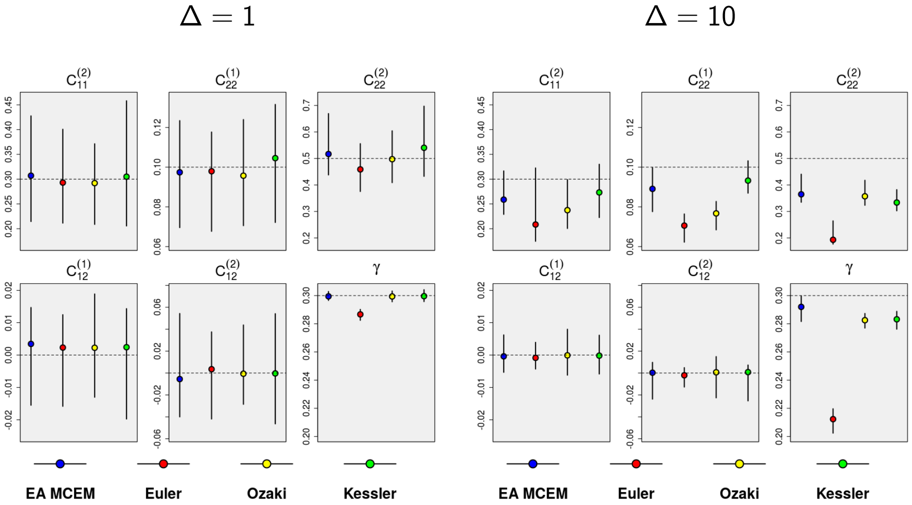


Results

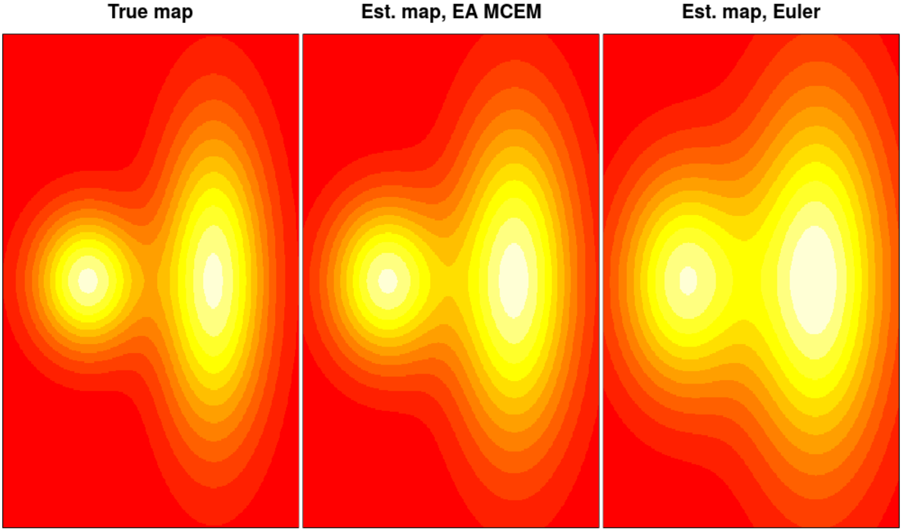


$\Delta = 1$
$\Delta = 10$
π_1
$\mu_1^{(1)}$
$\mu_1^{(2)}$
$\mu_2^{(1)}$
$\mu_2^{(2)}$
$C_{11}^{(1)}$
EA MCEM
Euler
Ozaki
Kessler

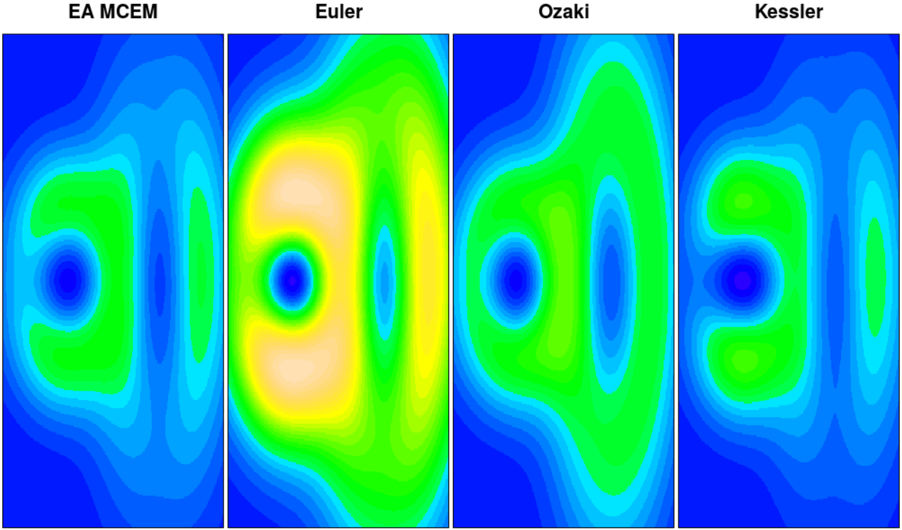
Results



Estimated map ($\Delta = 10$)



Estimation error ($\Delta = 10$)



Conclusions

On the example presented here:

- Euler method is the less robust method of all four;
- Pseudo likelihood methods seems robust:
  - Despite the fact that $n\Delta_n^p \nrightarrow 0$;
  - As easy as Euler method to implement;
- Exact approach seems robust:
  - Not supposed to depend on Δ;
  - However, computation time is much longer thant for other 3;
  - Harder to implement, more sensible to starting points.

Recommandations for movement ecology

- Ozaki or Kessler methods;
- Are more robust than Euler;
- Showed good robustness on our example;
- Easy implementation;
- Computation time do not depend on Δ.

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