Dual Approximate Dynamic Programming

P. Carpentier and J.-P. Chancelier and V. Leclère [1]

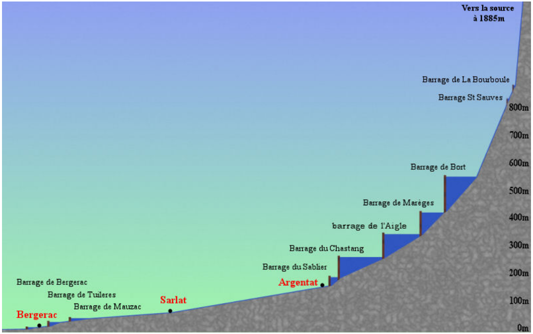
ENSTA ParisTech and ENPC ParisTech

[1] Joint work with J.-C. Alais,
supported by the FMJH Program Gaspard Monge for Optimization.

Lecture outline

Motivation

Electricity production management for hydro valleys



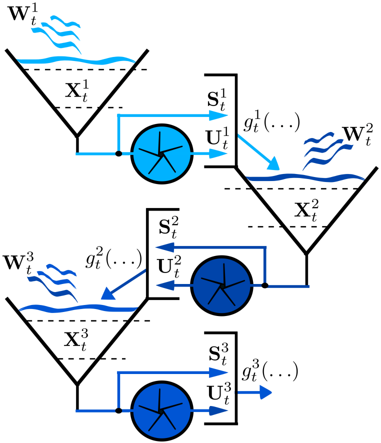
- *1 year time horizon*: compute each month the Bellman functions ("water values")
- *stochastic framework*: rain, market prices
- *large-scale valley*: 5 dams and more

We wish to remain within the scope of Dynamic Programming.

1. Dams management problem
    - Introduction
    - Hydro valley modeling
2. DADP in a nutshell
    - Decomposition Methods
    - Spatial Decomposition
    - Ideas behind DADP
3. Numerical experiments
    - Academic examples
    - More realistic examples

Mulstistage Stochastic Optimization: an Example

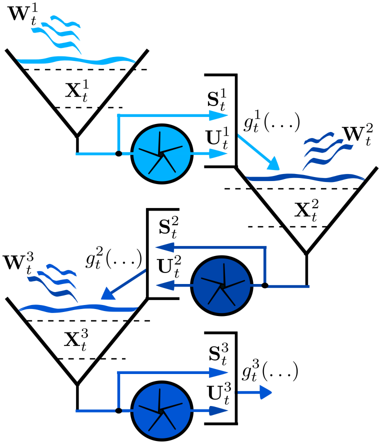
How to manage a chain of dam producing electricity from the turbine water to optimize the gain?



Mulststage Stochastic Optimization: an Example

How to manage a chain of dam producing electricity from the turbine water to optimize the gain?

$$\mathbb{E}\Big[\sum_{i=1}^{N}\sum_{t=0}^{T-1} L_t^i(\underbrace{X_t^i}_{\text{state}}, \underbrace{U_t^i}_{\text{control}}, \underbrace{W_{t+1}}_{\text{noise}})\Big]$$



Constraints:

- dynamics:
  $$X_{t+1} = f_t(X_t, U_t, W_{t+1}),$$
- nonanticipativity:
  $$U_t \preceq \mathcal{F}_t,$$
- spatial coupling:
  $$Z_t^{i+1} = g_t^i(X_t^i, U_t^i, W_{t+1}^i).$$

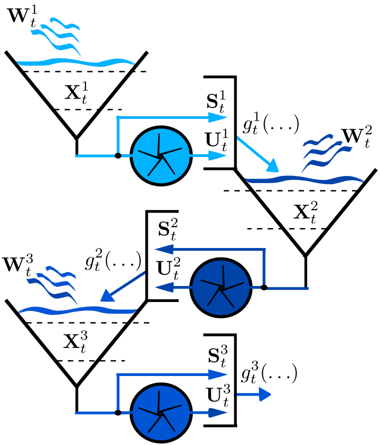
Mulstistage Stochastic Optimization: an Example

How to manage a chain of dam producing electricity from the turbine water to optimize the gain?

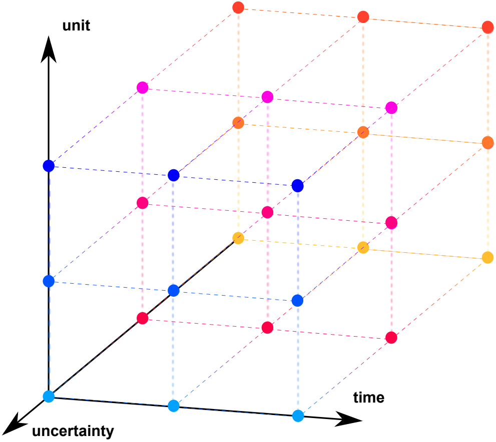
$$\mathbb{E}\Big[\sum_{i=1}^{N}\sum_{t=0}^{T-1} L_t^i(\underbrace{\boldsymbol{X}_t^i}_{\text{state}}, \underbrace{\boldsymbol{U}_t^i}_{\text{control}}, \underbrace{\boldsymbol{W}_{t+1}}_{\text{noise}})\Big]$$

Constraints:

- dynamics:
  $\boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{W}_{t+1})$,
- nonanticipativity:
  $\boldsymbol{U}_t \preceq \mathcal{F}_t$,
- spatial coupling:
  $\boldsymbol{Z}_t^{i+1} = g_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}^i)$.

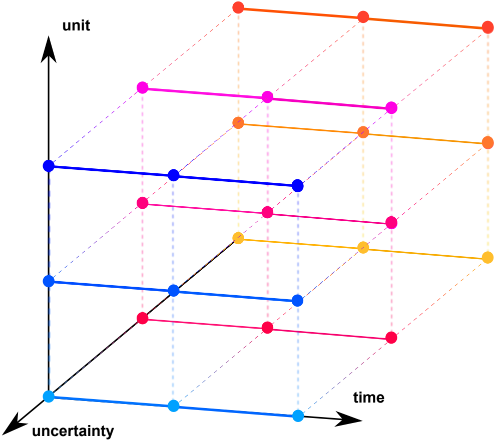


Couplings for Stochastic Problems



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

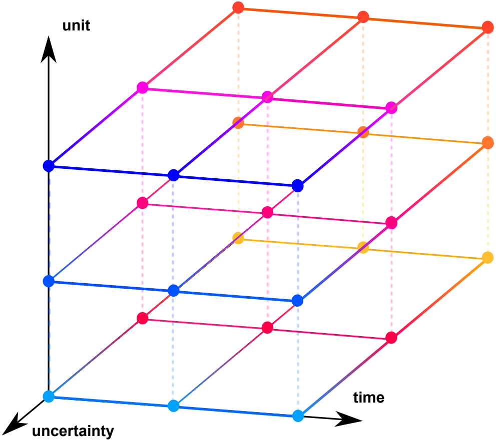
Couplings for Stochastic Problems: in Time



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

s.t. $\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$

Couplings for Stochastic Problems: in Uncertainty

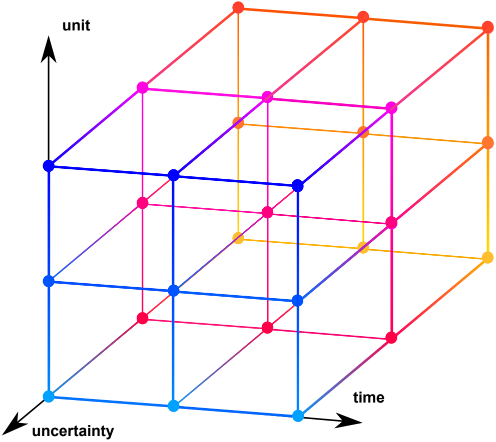


$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t = \sigma\big(\boldsymbol{W}_1, \dots, \boldsymbol{W}_t\big)$$

Couplings for Stochastic Problems: in Space



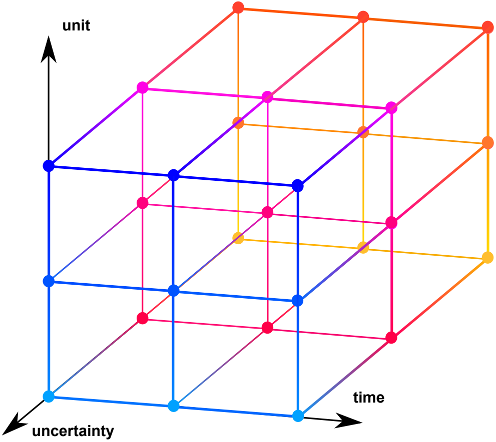
$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t = \sigma\big(\boldsymbol{W}_1, \dots, \boldsymbol{W}_t\big)$$

$$\sum_i \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$$

Couplings for Stochastic Problems: a Complex Problem



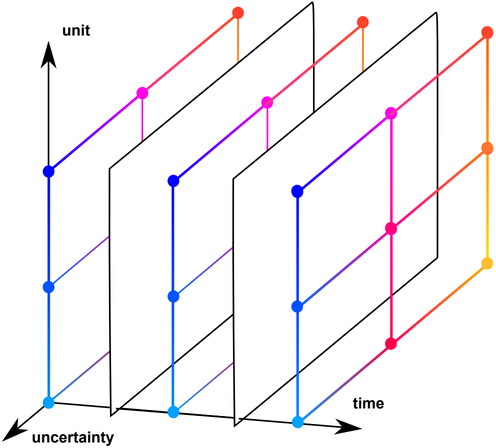
$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t = \sigma\big(\boldsymbol{W}_1, \dots, \boldsymbol{W}_t\big)$$

$$\sum_i \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$$

Decompositions for Stochastic Problems: in Time



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

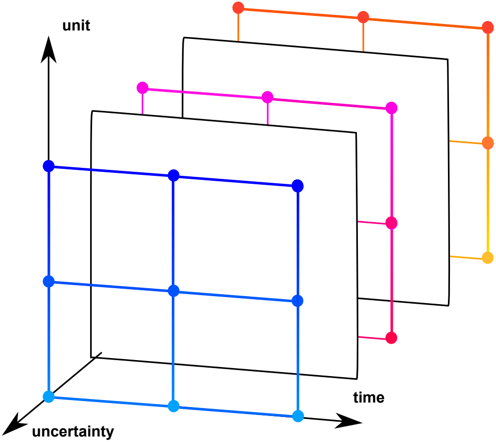
$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t = \sigma(\boldsymbol{W}_1, \dots, \boldsymbol{W}_t)$$

$$\sum_i \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$$

Dynamic Programming

Bellman (56)

Decompositions for Stochastic Problems: in Uncertainty



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

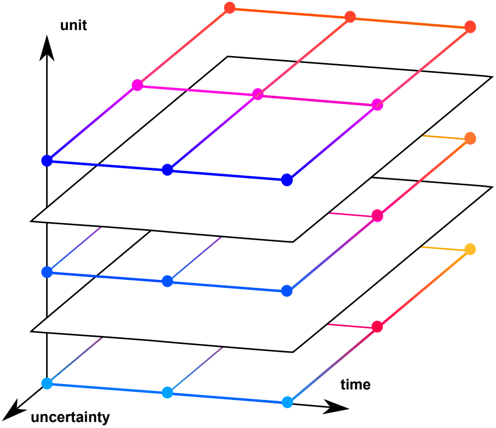
$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t = \sigma\big(\boldsymbol{W}_1, \dots, \boldsymbol{W}_t\big)$$

$$\sum_i \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$$

Progressive Hedging

Rockafellar - Wets (91)

Decompositions for Stochastic Problems: in Space



$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

$$\text{s.t.} \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1})$$

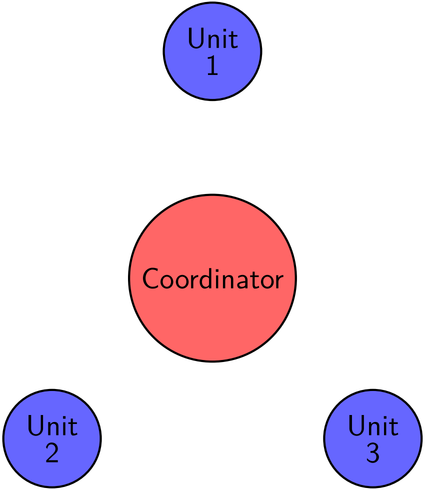
$$\boldsymbol{U}_t^i \preceq \mathcal{F}_t = \sigma\big(\boldsymbol{W}_1, \dots, \boldsymbol{W}_t\big)$$

$$\sum_i \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$$

Dual Approximate
Dynamic Programming

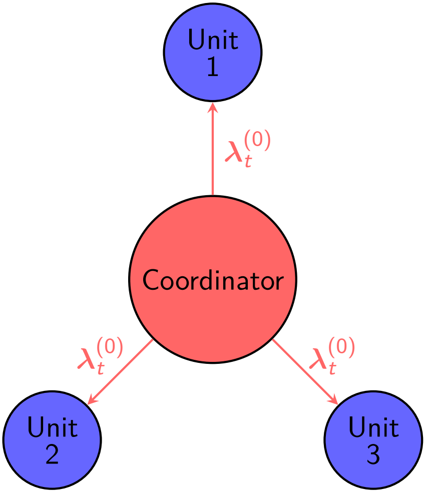
Intuition of Spatial Decomposition

- Satisfy a demand (over T time step) with N units of production at minimal cost.
- Price decomposition:
  - the coordinator sets a sequence of price λ_t,
  - the units send their production planning $\boldsymbol{U}_t^{(i)}$,
  - the coordinator compares total production and demand and updates the price,
  - and so on...



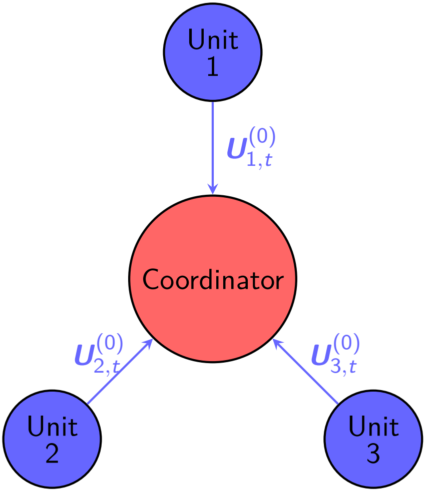
Intuition of Spatial Decomposition

- Satisfy a demand (over T time step) with N units of production at minimal cost.
- Price decomposition:
  - the coordinator sets a sequence of price λ_t,
  - the units send their production planning $U_t^{(i)}$,
  - the coordinator compares total production and demand and updates the price,
  - and so on...



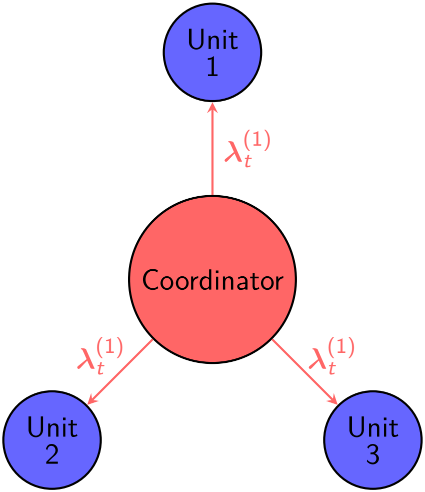
Intuition of Spatial Decomposition

- Satisfy a demand (over T time step) with N units of production at minimal cost.
- Price decomposition:
  - the coordinator sets a sequence of price λ_t,
  - the units send their production planning $\boldsymbol{U}_t^{(i)}$,
  - the coordinator compares total production and demand and updates the price,
  - and so on...



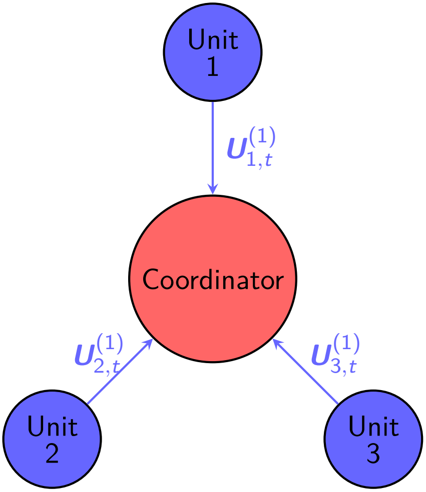
Intuition of Spatial Decomposition

- Satisfy a demand (over T time step) with N units of production at minimal cost.
- Price decomposition:
  - the coordinator sets a sequence of price λ_t,
  - the units send their production planning $\boldsymbol{U}_t^{(i)}$,
  - the coordinator compares total production and demand and updates the price,
  - and so on...



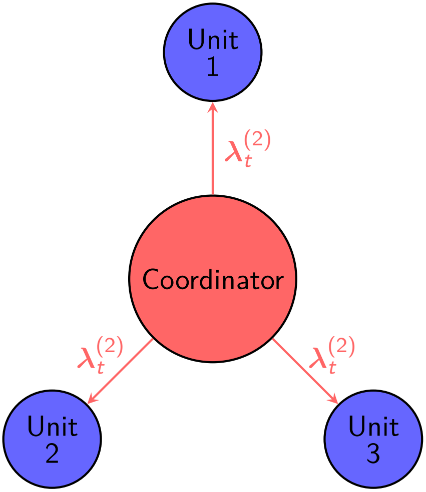
Intuition of Spatial Decomposition

- Satisfy a demand (over T time step) with N units of production at minimal cost.
- Price decomposition:
  - the coordinator sets a sequence of price λ_t,
  - the units send their production planning $\boldsymbol{U}_t^{(i)}$,
  - the coordinator compares total production and demand and updates the price,
  - and so on...



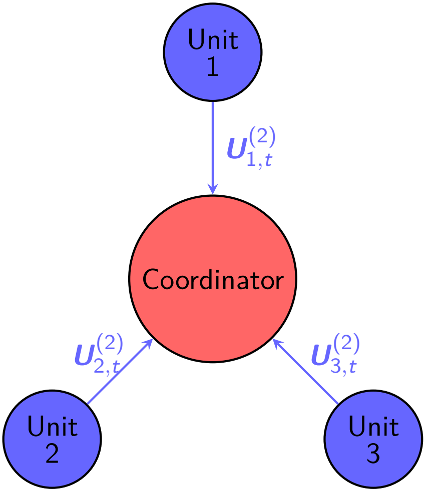
Intuition of Spatial Decomposition

- Satisfy a demand (over T time step) with N units of production at minimal cost.
- Price decomposition:
  - the coordinator sets a sequence of price λ_t,
  - the units send their production planning $\boldsymbol{U}_t^{(i)}$,
  - the coordinator compares total production and demand and updates the price,
  - and so on...

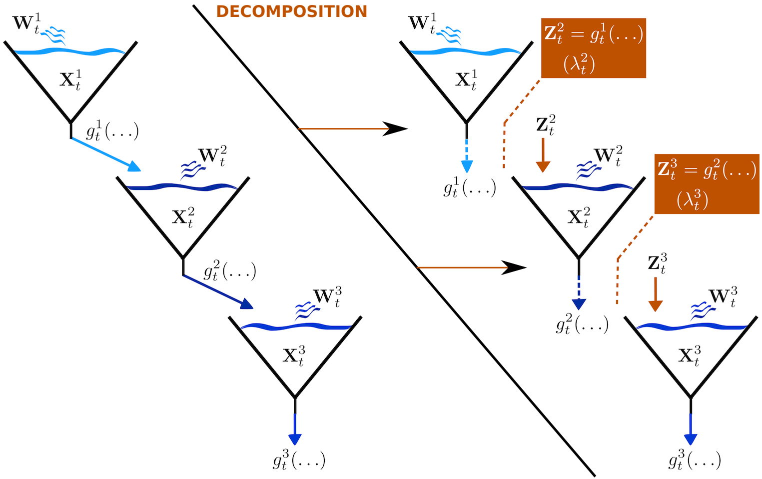


Intuition of Spatial Decomposition

- Satisfy a demand (over T time step) with N units of production at minimal cost.
- Price decomposition:
  - the coordinator sets a sequence of price λ_t,
  - the units send their production planning $\boldsymbol{U}_t^{(i)}$,
  - the coordinator compares total production and demand and updates the price,
  - and so on...



Application to dam management



Primal Problem

$$
\begin{aligned}
\min_{\boldsymbol{X},\boldsymbol{U}} \sum_{i=1}^{N} \; & \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i)\Big] \\
& \forall\, i, \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}), \quad \boldsymbol{X}_0^i = x_0^i, \\
& \forall\, i, \quad \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{ad}, \quad \boldsymbol{U}_t^i \preceq \mathcal{F}_t, \\
& \sum_{i=1}^{N} \theta_t^i(\boldsymbol{U}_t^i) = 0
\end{aligned}
$$

Solvable by DP with state $(\boldsymbol{X}_1, \dots, \boldsymbol{X}_N)$ (under noise independence assumption)

Primal Problem

$$\min_{\boldsymbol{X},\boldsymbol{U}} \sum_{i=1}^{N} \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + K^i(\boldsymbol{X}_T^i)\Big]$$

$$\forall\, i, \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}), \quad \boldsymbol{X}_0^i = x_0^i,$$

$$\forall\, i, \quad \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{ad}, \quad \boldsymbol{U}_t^i \preceq \mathcal{F}_t,$$

$$\sum_{i=1}^{N} \theta_t^i(\boldsymbol{U}_t^i) = 0 \qquad \rightsquigarrow \boldsymbol{\lambda}_t \quad \text{multiplier}$$

Solvable by DP with state $(\boldsymbol{X}_1, \dots, \boldsymbol{X}_N)$ (under noise independence assumption)

Primal Problem with Dualized Constraint

$$\min_{\boldsymbol{X},\boldsymbol{U}} \max_{\boldsymbol{\lambda}} \sum_{i=1}^{N} \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \langle \boldsymbol{\lambda}_t, \theta_t^i(\boldsymbol{U}_t^i) \rangle + K^i(\boldsymbol{X}_T^i)\Big]$$

$$\forall\, i, \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}), \quad \boldsymbol{X}_0^i = x_0^i,$$

$$\forall\, i, \quad \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{ad}, \quad \boldsymbol{U}_t^i \preceq \mathcal{F}_t,$$

Coupling constraint dualized $\Longrightarrow$ remaining constraints are i by i

Dual Problem

$$\max_{\lambda} \min_{\boldsymbol{X},\boldsymbol{U}} \sum_{i=1}^{N} \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \langle \boldsymbol{\lambda}_t, \theta_t^i(\boldsymbol{U}_t^i)\rangle + K^i(\boldsymbol{X}_T^i)\Big]$$

$$\forall\, i, \quad \boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}), \quad \boldsymbol{X}_0^i = x_0^i,$$

$$\forall\, i, \quad \boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{ad}, \quad \boldsymbol{U}_t^i \preceq \mathcal{F}_t,$$

Exchange operator min and max to obtain a new problem

Decomposed Dual Problem

$$\max_{\lambda} \sum_{i=1}^{N} \min_{\boldsymbol{X}^i, \boldsymbol{U}^i} \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \langle \boldsymbol{\lambda}_t, \theta_t^i(\boldsymbol{U}_t^i) \rangle + K^i(\boldsymbol{X}_T^i) \Big]$$

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}), \quad \boldsymbol{X}_0^i = x_0^i,$$

$$\boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{ad}, \quad \boldsymbol{U}_t^i \preceq \mathcal{F}_t,$$

For a given λ, minimum of sum is sum of minima

Inner Minimization Problem

$$\min_{\boldsymbol{X}^i, \boldsymbol{U}^i} \ \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \langle \boldsymbol{\lambda}_t, \theta_t^i(\boldsymbol{U}_t^i)\rangle + K^i(\boldsymbol{X}_T^i)\Big]$$

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}), \quad \boldsymbol{X}_0^i = x_0^i,$$

$$\boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{ad}, \quad \boldsymbol{U}_t^i \preceq \mathcal{F}_t,$$

We have N smaller subproblems. Can they be solved by DP?

Inner Minimization Problem

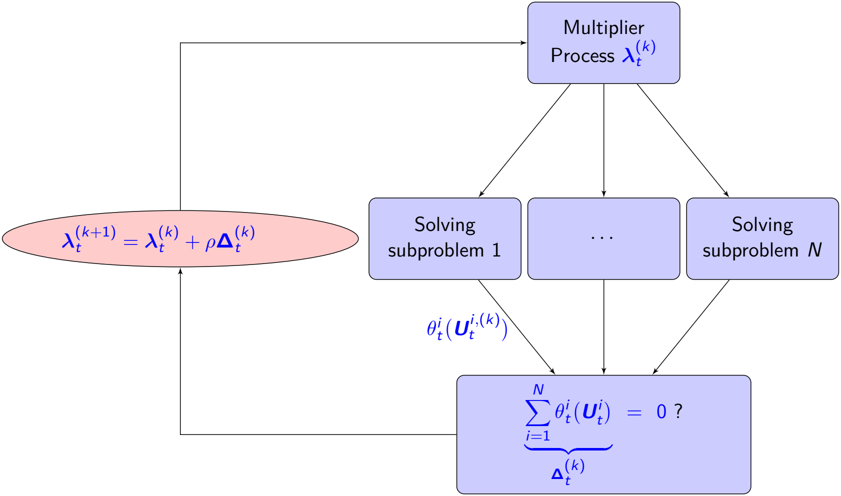
$$\min_{\boldsymbol{X}^i,\boldsymbol{U}^i} \; \mathbb{E}\Big[\sum_{t=0}^{T} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}) + \langle \boldsymbol{\lambda}_t, \theta_t^i(\boldsymbol{U}_t^i)\rangle + K^i(\boldsymbol{X}_T^i)\Big]$$

$$\boldsymbol{X}_{t+1}^i = f_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}), \quad \boldsymbol{X}_0^i = x_0^i,$$

$$\boldsymbol{U}_t^i \in \mathcal{U}_{t,i}^{ad}, \quad \boldsymbol{U}_t^i \preceq \mathcal{F}_t,$$

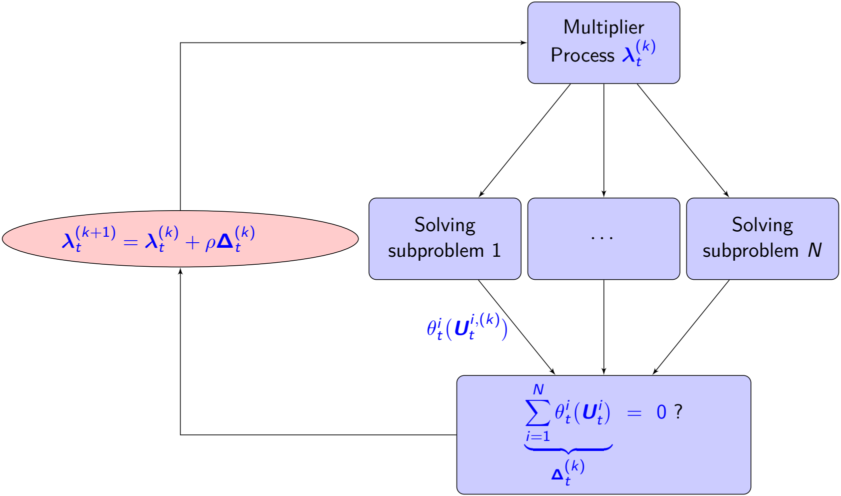
No: $\boldsymbol{\lambda}$ is a time-dependent noise $\rightsquigarrow$ $\boldsymbol{X}_t^i$ is not a proper state, but rather $(\boldsymbol{W}_1, \dots, \boldsymbol{W}_t)$

Stochastic spatial decomposition scheme



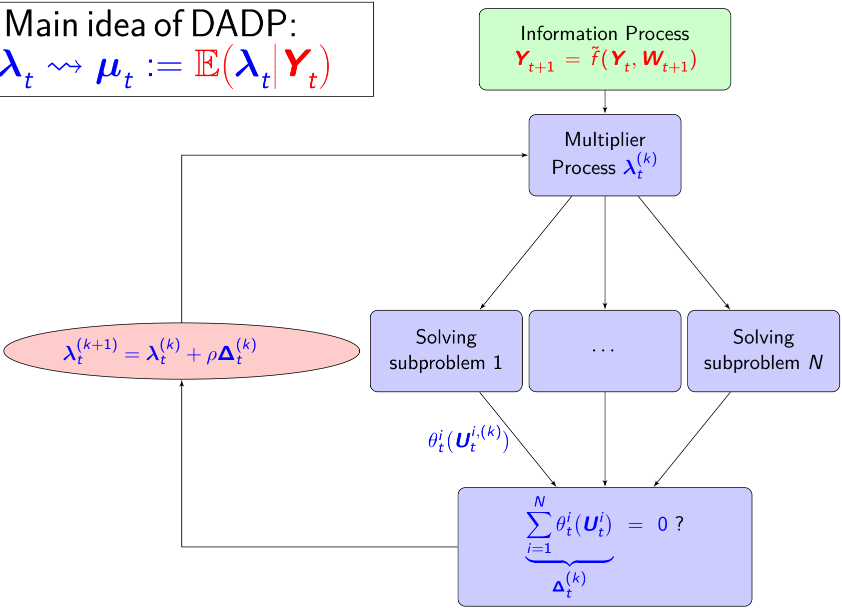
Main idea of DADP:

$$\boldsymbol{\lambda}_t \rightsquigarrow \boldsymbol{\mu}_t := \mathbb{E}\big(\boldsymbol{\lambda}_t \big| \boldsymbol{Y}_t\big)$$



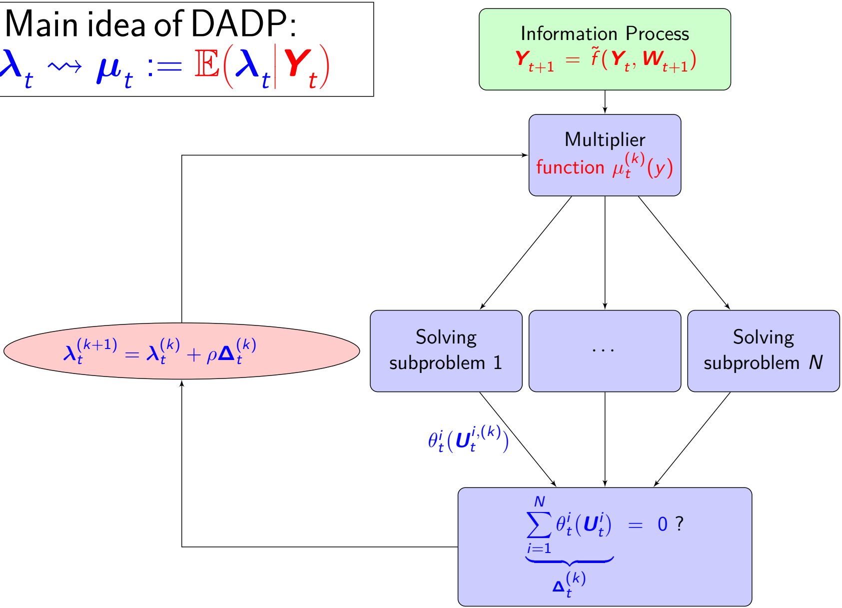
Main idea of DADP:

$$\boldsymbol{\lambda}_t \rightsquigarrow \boldsymbol{\mu}_t := \mathbb{E}\big(\boldsymbol{\lambda}_t \big| \boldsymbol{Y}_t\big)$$



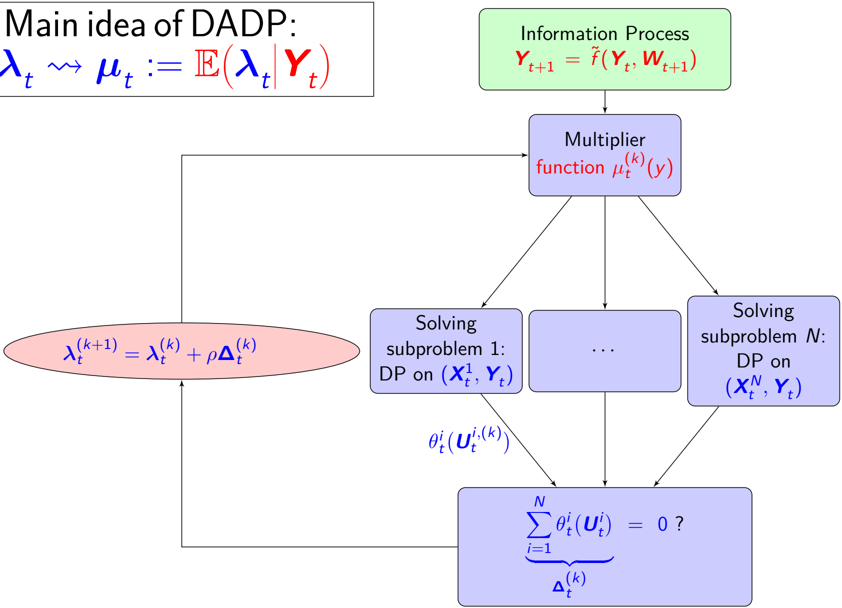
Main idea of DADP:

$$\boldsymbol{\lambda}_t \rightsquigarrow \boldsymbol{\mu}_t := \mathbb{E}\big(\boldsymbol{\lambda}_t \big| \boldsymbol{Y}_t\big)$$



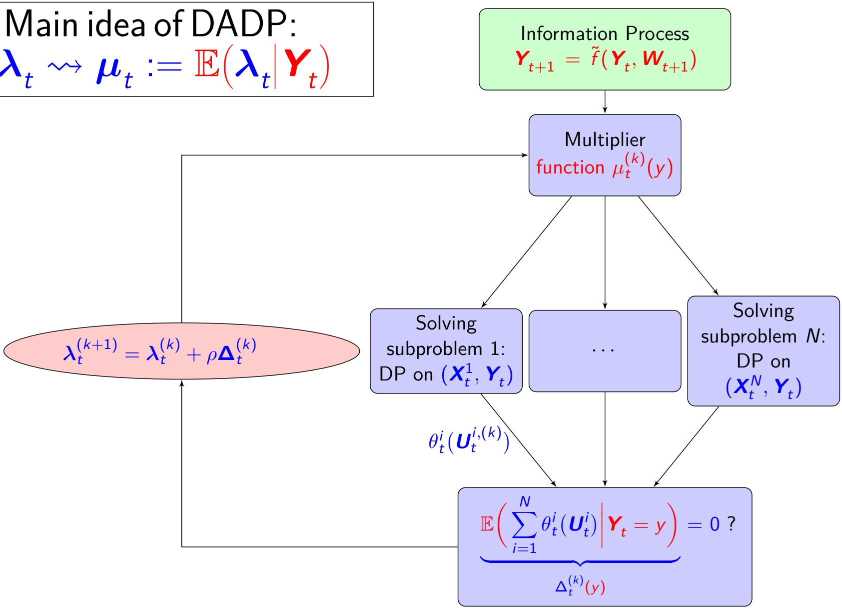
Main idea of DADP:

$$\boldsymbol{\lambda}_t \rightsquigarrow \boldsymbol{\mu}_t := \mathbb{E}\big(\boldsymbol{\lambda}_t \big| \boldsymbol{Y}_t\big)$$



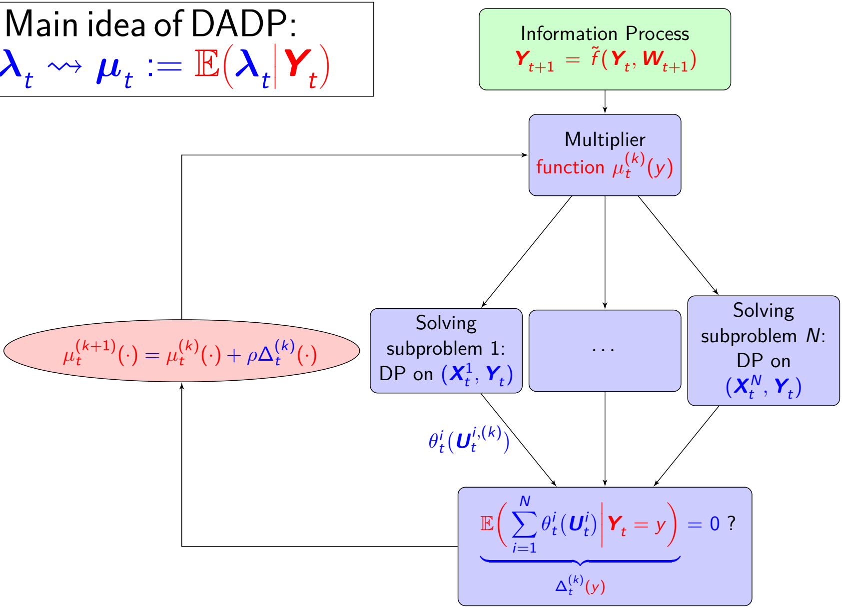
Main idea of DADP:

$$\boldsymbol{\lambda}_t \rightsquigarrow \boldsymbol{\mu}_t := \mathbb{E}\big(\boldsymbol{\lambda}_t \big| \boldsymbol{Y}_t\big)$$

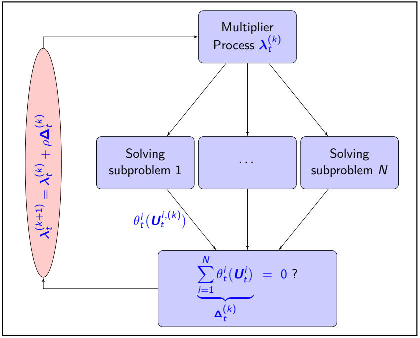


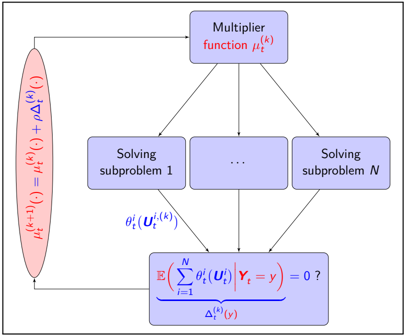
Main idea of DADP:

$$\boldsymbol{\lambda}_t \rightsquigarrow \boldsymbol{\mu}_t := \mathbb{E}\big(\boldsymbol{\lambda}_t \big| \boldsymbol{Y}_t\big)$$



Main idea of DADP: $\boldsymbol{\lambda}_t \rightsquigarrow \boldsymbol{\mu}_t := \mathbb{E}\Big(\boldsymbol{\lambda}_t \Big| \boldsymbol{Y}_t\Big)$





Main problems:

- Subproblems not easily solvable by DP
- $\boldsymbol{\lambda}^{(k)}$ live in a huge space

Advantages:

- Subproblems solvable by DP with state $\big(\boldsymbol{X}_t^i, \boldsymbol{Y}_t\big)$
- $\mu^{(k)}$ live in a smaller space

Dual approximation as constraint relaxation

The original problem is (abstract form)

$$\begin{aligned} \min_{\boldsymbol{U}\in\mathcal{U}} \quad & J(\boldsymbol{U}) \\ s.t. \quad & \Theta(\boldsymbol{U}) = 0 \end{aligned}$$

written as

$$\min_{\boldsymbol{U}\in\mathcal{U}} \max_{\boldsymbol{\lambda}} \quad J(\boldsymbol{U}) + \mathbb{E}\big[\langle \boldsymbol{\lambda}, \Theta(\boldsymbol{U})\rangle\big]$$

Subsituting $\boldsymbol{\lambda}$ by $\mathbb{E}\big(\boldsymbol{\lambda}\big|\boldsymbol{Y}\big)$ gives

$$\min_{\boldsymbol{U}\in\mathcal{U}} \max_{\boldsymbol{\lambda}} \quad J(\boldsymbol{U}) + \mathbb{E}\Big[\Big\langle \mathbb{E}\big(\boldsymbol{\lambda}\big|\boldsymbol{Y}\big), \Theta(\boldsymbol{U})\Big\rangle\Big]$$

Dual approximation as constraint relaxation

The original problem is (abstract form)

$$\min_{\boldsymbol{U}\in\mathcal{U}} \quad J(\boldsymbol{U})$$
$$s.t. \quad \Theta(\boldsymbol{U}) = 0$$

written as

$$\min_{\boldsymbol{U}\in\mathcal{U}} \max_{\boldsymbol{\lambda}} \quad J(\boldsymbol{U}) + \mathbb{E}\big[\langle \boldsymbol{\lambda}, \Theta(\boldsymbol{U})\rangle\big]$$

Subsituting $\boldsymbol{\lambda}$ by $\mathbb{E}(\boldsymbol{\lambda} \mid \boldsymbol{Y})$ gives

$$\min_{\boldsymbol{U}\in\mathcal{U}} \max_{\boldsymbol{\lambda}} \quad J(\boldsymbol{U}) + \mathbb{E}\Big[\big\langle \boldsymbol{\lambda}, \mathbb{E}\big(\Theta(\boldsymbol{U}) \big| \boldsymbol{Y}\big)\big\rangle\Big]$$

Dual approximation as constraint relaxation

The original problem is (abstract form)

$$\min_{\boldsymbol{U}\in\mathcal{U}} \quad J(\boldsymbol{U}) \qquad s.t. \quad \Theta(\boldsymbol{U}) = 0$$

written as

$$\min_{\boldsymbol{U}\in\mathcal{U}} \max_{\boldsymbol{\lambda}} \quad J(\boldsymbol{U}) + \mathbb{E}\big[\langle \boldsymbol{\lambda}, \Theta(\boldsymbol{U})\rangle\big]$$

Subsituting $\boldsymbol{\lambda}$ by $\mathbb{E}\big(\boldsymbol{\lambda}\big|\boldsymbol{Y}\big)$ gives

$$\min_{\boldsymbol{U}\in\mathcal{U}} \max_{\boldsymbol{\lambda}} \quad J(\boldsymbol{U}) + \mathbb{E}\Big[\Big\langle \boldsymbol{\lambda}, \mathbb{E}\big(\Theta(\boldsymbol{U})\big|\boldsymbol{Y}\big)\Big\rangle\Big]$$

equivalent to

$$\min_{\boldsymbol{U}\in\mathcal{U}} \quad J(\boldsymbol{U}) \qquad s.t. \quad \mathbb{E}\big(\Theta(\boldsymbol{U})\big|\boldsymbol{Y}\big) = 0$$

Three Interpretations of DADP

- DADP as an approximation of the optimal multiplier

$$\boldsymbol{\lambda}_t \quad \rightsquigarrow \quad \mathbb{E}\big(\boldsymbol{\lambda}_t \big| \boldsymbol{Y}_t\big) \ .$$

- DADP as a decision-rule approach in the dual

$$\max_{\boldsymbol{\lambda}} \min_{\boldsymbol{U}} L(\boldsymbol{\lambda}, \boldsymbol{U}) \quad \rightsquigarrow \quad \max_{\boldsymbol{\lambda}_t \preceq \boldsymbol{Y}_t} \min_{\boldsymbol{U}} L(\boldsymbol{\lambda}, \boldsymbol{U}) \ .$$

- DADP as a constraint relaxation in the primal

$$\sum_{i=1}^{n} \theta_t^i(\boldsymbol{U}_t^i) = 0 \quad \rightsquigarrow \quad \mathbb{E}\bigg(\sum_{i=1}^{n} \theta_t^i(\boldsymbol{U}_t^i) \bigg| \boldsymbol{Y}_t\bigg) = 0 \ .$$

Theoretical Results

- Consistence of the approximation (if we consider a sequence of approximated problems).
- Existence of multiplier of the coupling constraint.
- Convergence of the decomposition algorithm for a given relaxation.
- Lower and upper bounds on the original problem.
- A posteriori verification allowing for better multiplier update.

Consistence of the Approximation Scheme

- The DADP algorithm solves a relaxation $(\mathcal{P}_{\boldsymbol{Y}})$ of the original problem $(\mathcal{P})$ where

$$\sum_{i=1}^{n} \theta_t^i(\boldsymbol{U}_t^i) = 0 \qquad \rightsquigarrow \qquad \mathbb{E}\Big(\sum_{i=1}^{n} \theta_t^i(\boldsymbol{U}_t^i) \Big| \boldsymbol{Y}_t \Big) = 0$$

$(\mathcal{P}_{\boldsymbol{Y}})$.

- Question: if we consider a sequence of information processes $\{\boldsymbol{Y}^{(n)}\}_{n\in\mathbb{N}}$, such that the information converges

$$\sigma\big(\boldsymbol{Y}_t^{(n)}\big) \to \sigma\big(\boldsymbol{W}_0, \cdots, \boldsymbol{W}_t\big)$$

does the associated sequence $\big(\boldsymbol{U}^{\boldsymbol{Y}^{(n)}}\big)$ of optimal control converges toward an optimal control of $(\mathcal{P})$?

Epiconvergence of Approximation

Epiconvergence result

Assume that

- the cost functions L_t^i, dynamic functions f_t^i and constraint functions θ_t^i are continuous;
- the noise variables $\boldsymbol{W}_t$ are essentially bounded;
- the constraint sets $\mathcal{U}_{i,t}^{\mathrm{ad}}$ are bounded.

Consider a sequence of information process $\left\{\boldsymbol{Y}^{(n)}\right\}_{n\in\mathbb{N}}$ such that $\sigma\left(\boldsymbol{Y}^{(n)}\right)$ Kudo-converges toward $\mathcal{F}_\infty$. Let $\boldsymbol{U}^{(n)}$ be an ε_n-optimal solution to the relaxed problem $\left(\mathcal{P}^{\boldsymbol{Y}^{(n)}}\right)$.

Then, every cluster point[a] of $\left\{\boldsymbol{U}^{(n)}\right\}_{n\in\mathbb{N}}$ is an optimal solution of the relaxation corresponding to $\mathcal{F}_\infty$.

[a] for the topology of the convergence in probability

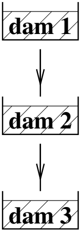
Choosing an Information Process $\boldsymbol{Y}$

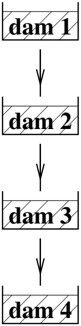
- Perfect memory: $\boldsymbol{Y}_t^i = (\boldsymbol{W}_0, \dots, \boldsymbol{W}_t)$.
  $\rightsquigarrow$ equivalent to original problem, no numerical gain.
- Minimal information: $\boldsymbol{Y}_t^i \equiv \mathrm{cste}$.
  $\rightsquigarrow$ equivalent to replacing a.s. constraint by expected constraint. Subproblems solved efficiently (state $\boldsymbol{X}_t^i$), multiplier is deterministic.
- Static information: $\boldsymbol{Y}_t^i = h_t^i(\boldsymbol{W}_t)$.
  $\rightsquigarrow$ Subproblems solved efficiently (state $\boldsymbol{X}_t^i$).
- Dynamic information: $\boldsymbol{Y}_{t+1}^i = h_t^i(\boldsymbol{Y}_t^i, \boldsymbol{W}_{t+1})$.
  $\rightsquigarrow$ A number of possibilities. Some ideas:
  - mimicking the trajectory of the state of another unit (phantom state),
  - mimicking the control of other units,
  - Markov chain representing rougly the general state of the system.

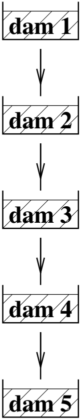
Numerical Advantages of a finitely supported $\boldsymbol{Y}$

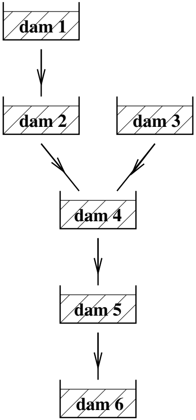
- Assume that each noise $\boldsymbol{W}_t$ take w values, and the constraint function take value in $\mathbb{R}$.
- Then the multiplier $\boldsymbol{\lambda}_t$ of the almost sure constraint at time t lives in $\mathbb{R}^{wt}$.
- Assume that the information process at time t take y values, then the multiplier of the relaxed constraint $\boldsymbol{\mu}_t$ lives in $\mathbb{R}^{y}$.
- Moreover each subproblems take "only" roughly y times more computational effort to solve than the subproblem with local state $\boldsymbol{X}_t^i$.

Four case studies









Discretization

$T \rightsquigarrow 12$

$\boldsymbol{X} \rightsquigarrow 41$

$\boldsymbol{U} \rightsquigarrow 6$

$\boldsymbol{W} \rightsquigarrow 10$

Results

Valley	**3-Dams**	**4-Dams**	**5-Dams**	**6-Dams**
DP CPU time	*5*'	*1630*'	*461200*'	N.A.
DP value	2482.0	3742.7	4681.6	N.A.
$\mathrm{SDDP_d}$ value	2474.2	3736.4	4672.2	7014.8
$\mathrm{SDDP_d}$ CPU time	*0.3*'	*2*'	*16*'	*320*'
$\mathrm{SDDP_c}$ value	2479.1	3739.7	4678.5	7016.4
$\mathrm{SDDP_c}$ CPU time	*5*'	*9*'	*11*'	*13*'

Table: Results obtained by DP, $\mathrm{SDDP_d}$ and $\mathrm{SDDP_c}$

Valley	3-Dams	4-Dams	5-Dams	6-Dams
DADP CPU time	3'	6'	5'	13'
DADP value	2401.3	3667.0	4633.7	6816.5
Gap with DP	−3.2%	−2.0%	−1.0%	−2.8%

Table: Results obtained by DADP "Expectation"

Results obtained using a 4 cores – 8 threads Intel®Core i7 based computer.

Results

Valley	3-Dams	4-Dams	5-Dams	6-Dams
DP CPU time	*5'*	*1630'*	*461200'*	N.A.
DP value	2482.0	3742.7	4681.6	N.A.
$\mathrm{SDDP_d}$ value	2474.2	3736.4	4672.2	7014.8
$\mathrm{SDDP_d}$ CPU time	*0.3'*	*2'*	*16'*	*320'*
$\mathrm{SDDP_c}$ value	2479.1	3739.7	4678.5	7016.4
$\mathrm{SDDP_c}$ CPU time	*5'*	*9'*	*11'*	*13'*

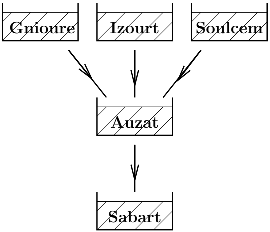
Table: Results obtained by DP, $\mathrm{SDDP_d}$ and $\mathrm{SDDP_c}$

Valley	3-Dams	4-Dams	5-Dams	6-Dams
DADP CPU time	*3'*	*6'*	*5'*	*13'*
DADP value	2401.3	3667.0	4633.7	6816.5
Gap with DP	$-\mathbf{3.2\%}$	$-\mathbf{2.0\%}$	$-\mathbf{1.0\%}$	$-\mathbf{2.8\%}$

Table: Results obtained by DADP "Expectation"

Results obtained using a 4 cores – 8 threads Intel®Core i7 based computer.

Two "true" valleys

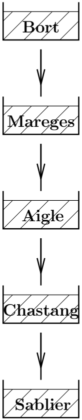


Discretization

$T \rightsquigarrow 12$, $\boldsymbol{W} \rightsquigarrow 10$

realistic grids for $\boldsymbol{U}$ and $\boldsymbol{X}$

Vicdessos



Dordogne

Results

Valley	**Vicdessos**	**Dordogne**
$\mathrm{SDDP_d}$ CPU time	*90*'	*86000*'
$\mathrm{SDDP_d}$ value	2232.1	21904.5
$\mathrm{SDDP_c}$ CPU time	*23*'	*28*'
$\mathrm{SDDP_c}$ value	2220.6	22035.2

Table: Results obtained by $\mathrm{SDDP_d}$ and $\mathrm{SDDP_c}$

Valley	**Vicdessos**	**Dordogne**
DADP CPU time	*10*'	*155*'
DADP value	2237.4	21499.8
Gap with $\mathrm{SDDP_d}$	$+0.2\%$	-1.8%

Table: Results obtained by DADP "Expectation"

Results

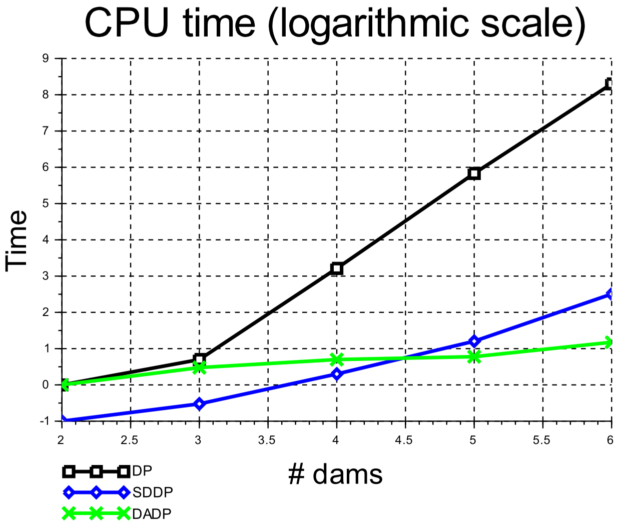
Valley	**Vicdessos**	**Dordogne**
$\mathrm{SDDP_d}$ CPU time	*90'*	*86000'*
$\mathrm{SDDP_d}$ value	2232.1	21904.5
$\mathrm{SDDP_c}$ CPU time	*23'*	*28'*
$\mathrm{SDDP_c}$ value	2220.6	22035.2

Table: Results obtained by $\mathrm{SDDP_d}$ and $\mathrm{SDDP_c}$

Valley	**Vicdessos**	**Dordogne**
DADP CPU time	*10'*	*155'*
DADP value	2237.4	21499.8
Gap with $\mathrm{SDDP_d}$	$+\mathbf{0.2\%}$	$-\mathbf{1.8\%}$

Table: Results obtained by DADP "Expectation"

CPU time comparison



Conclusions and perspectives

Conclusions for DADP

- Fast numerical convergence of the method.
- Near-optimal results even when using a “crude” relaxation.
- Method that can be used for very large valleys

General perspectives

- Apply to more complex topologies (smart grids).
- Connection with other decomposition methods.
- Theoretical study.

Conclusions and perspectives

Conclusions for DADP

- Fast numerical convergence of the method.
- Near-optimal results even when using a “crude” relaxation.
- Method that can be used for very large valleys

General perspectives

- Apply to more complex topologies (smart grids).
- Connection with other decomposition methods.
- Theoretical study.

P. Carpentier et G. Cohen.
Décomposition-coordination en optimisation déterministe et stochastique.
En préparation, Springer, 2016.

P. Girardeau.
Résolution de grands problèmes en optimisation stochastique dynamique.
Thèse de doctorat, Université Paris-Est, 2010.

J.-C. Alais.
Risque et optimisation pour le management d'énergies.
Thèse de doctorat, Université Paris-Est, 2013.

V. Leclère.
Contributions aux méthodes de décomposition en optimisation stochastique.
Thèse de doctorat, Université Paris-Est, 2014.

K. Barty, P. Carpentier, G. Cohen and P. Girardeau,
Price decomposition in large-scale stochastic optimal control.
arXiv, math.OC 1012.2092, 2010.