Sparse recovery under weak moment assumptions

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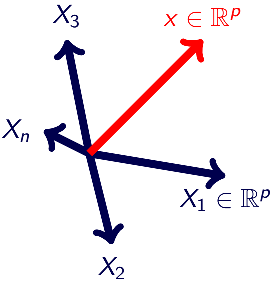
joint works with Sjoerd Dirksen, Shahar Mendelson and Holger Rauhut

Exact reconstruction from few linear measurements (Compressed sensing)



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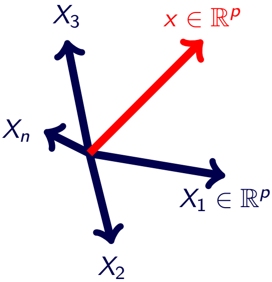

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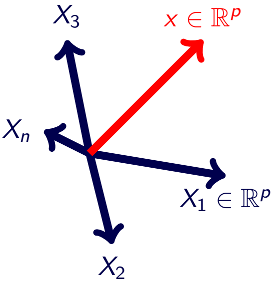

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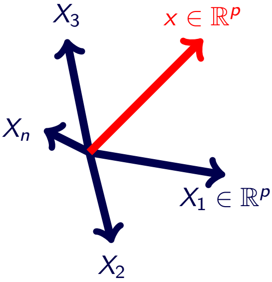

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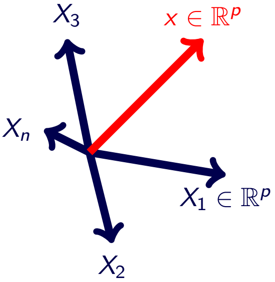

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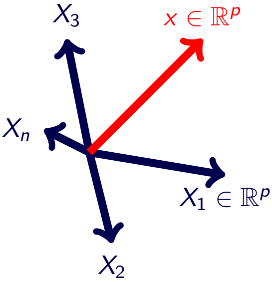

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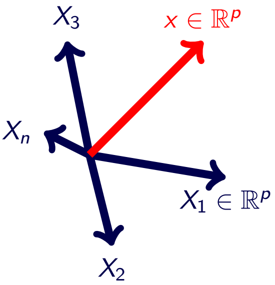

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p : space dimension, n : number of measurements, s : sparsity parameter.

ℓ_0-minimization is NP-hard

ℓ_0-minimization procedure

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3. Natarajan, 1995 : ℓ_0-minimization is NP-hard (solves the "exact cover by 3-sets problem").

Basis pursuit – [Logan, 1965], [Donoho, Logan, 1992], [...]

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Question : construction of Γ satisfying ER(s) with $n \sim s\log(ep/s)$.

RIP($c_0 s$) : for any $\|x\|_0 \leq c_0 s, \quad \frac{1}{2}\|x\|_2 \leq \|\Gamma x\|_2 \leq \frac{3}{2}\|x\|_2.$

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$(\Leftrightarrow \Gamma B_1^p$ is $s -$ neighborly, [Donoho, 05])

$(\Leftarrow$ Incoherency conditions)

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3. Structured matrices : partial Fourier matrices satisfy RIP(s) [Rudelson, Vershynin, Candès, Tao, Bourgain] when $n \gtrsim s\log^3(p)$.

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Here : we will need $\log p$ moments for the coordinates x_j's.

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Exact reconstruction under weak concentration property cannot be studied via RIP

A weaker condition than RIP

Proposition (L. and Mendelson)

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Then, $\forall t \in \sqrt{c_1 s} B_1^p \cap S_2^{p-1}$, $\|\Gamma t\|_2 \ge c_2 > 0$ *and so* *ER(s)* *holds.*

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- RHS of RIP requires deviation (ψ_2).

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Here, we assume that the measurement vector $X = (x_1, \ldots, x_p) \in \mathbb{R}^p$ is such that : $\|x_j\|_{L_2} = 1$ and

$$\|x_j\|_{L_q} \leq \kappa_0 q^\eta, \text{ for } q \sim \log(p)$$

for some $\eta \geq 1/2$.

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[RV13] : "Small ball probabilities for linear images of high dimensional distributions" M. Rudelson and R. Vershynin.

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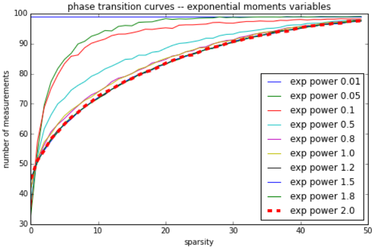
Γ does not satisfy the ER(1) with probability at least $1/2$.

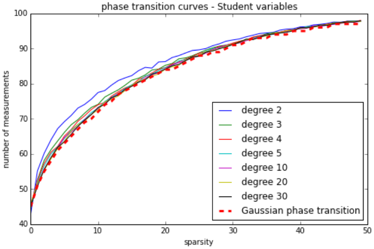
$\Rightarrow$ We need at least $\log p / \log\log p$ moments for exact reconstruction via basis pursuit.

phase transition diagram for Exponential and Student variables.

ψ_γ variables $\mathrm{sign}(g)|g|^{2/\gamma}$ where $g \sim \mathcal{N}(0,1)$



Student variables of degree k density $\sim (1+t^2)^{-(k+1)/2}$



A price to pay for convex relaxation

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Then, with probability at least $1 - 2\exp(-c_1 n \beta_0^2)$,

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$\Rightarrow$ We don't need moment assumption for $\ell_0 -$ *minimization*. This proves that there is a *price to pay* in terms of concentration for convex relaxation.

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4. $\log p / \log\log p$ moments is a necessary price to pay for convex relaxation.

Data : $y = Q_\theta(\Gamma x)$ where $Q_\theta : \mathbb{R}^m \to (\theta\mathbb{Z} + \theta/2)^m$.

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Results for BPDN_q from Jacques, Hammond, and Fadili, *Dequantizing compressed sensing : when oversampling and non-Gaussian constraints combine*, when $q \to \infty$ shows that if

$$n \gtrsim \Big(s \log(ep/s)\Big)^{q/2}$$

then $(RIP)_{q,2}$ holds with large probability for Gaussian measurements matrices Γ and then for any x

$$\|\hat{x}_{BPDN_q} - x\|_2 \lesssim \frac{\sigma_s(x)_1}{\sqrt{s}} + \frac{\theta}{\sqrt{q+1}}$$

Application to quantized CS

Theorem (Dirksen, L. and Rauhut)

For Gaussian measurements, when $n \gtrsim s\log(ep/s)$ *then*

$$\|\hat{x}_{BPDN_\infty} - x\|_2 \lesssim \frac{\sigma_s(x)_1}{\sqrt{s}} + \theta.$$

Moreover, $\hat{x}_{BPDN_\infty}$ *is quantized consistent :* $y = Q_\theta(\Gamma \hat{x}_{BPDN_\infty})$.

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For the quantization problem $RIP_{q,2}$ requires more measurements.
For analysis based on $RIP_{q,r}$: for any $x \in \Sigma_s$

$$c\|x\|_r \leq \|\Gamma x\|_q \leq C\|x\|_r$$

two phenomena occur :

1. more measurements than $s\log(ep/s)$
2. other type of matrices than Gaussian (adjacency matrices, stable processes).

Bypassing the RIP based approach show that none of these two phenomena actually occur : one can use $s\log(ep/s)$ Gaussian measures.

Thanks for your attention

G. Lecué and S. Mendelson, *Sparse recovery under weak moment assumption*. To appear in *Journal of the European Mathematical society*, Jan. 2014.

S. Dirksen, G. Lecué and H. Rauhut, *On the gap between restricted isometry properties and sparse recovery conditions*. To appear in *IEEE Transactions on Information Theory*, March 2015.

Sparse Linear Regression

Noisy data - LASSO

Data : $y = \mathcal{X}\beta^* + \sigma W$ where $W \sim \mathcal{N}_p(0, I_n)$ and

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where

$$\|\beta\|_{n,1} = \sum_{j=1}^{p} r_{n,j} |\beta_j| \text{ and } r_{n,j} = \left(\frac{\mathcal{X}^\top \mathcal{X}}{n} \right)_{jj}.$$

Restricted eigenvalue condition : For any $I \subset [p]$ s.t. $|I| \leq s$, $v \in \mathbb{R}^p$

$$\|v_{I^c}\|_1 \leq 3\|v_I\|_1 \quad \Rightarrow \quad \|\Gamma v\|_2 \geq \kappa(s)\|v_I\|_2 \qquad (REC(s))$$

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Remark : (Null space property) $\forall I \subset [p]$ s.t. $|I| \leq s$, $v \in \mathbb{R}^p$

$$\|v_{I^c}\|_1 < \|v_I\|_1 \quad \Rightarrow \quad \|\Gamma v\|_2 > 0 \qquad (NSP(s))$$

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Estimation under REC via the LASSO - [BRT,07]

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2. the same is true for the Compatibility Condition of S. van de Geer
3. the same is true for normalized measurement matrices.

Problem

Given a collection $\{C_j : j \in [p]\}$ of 3-element subsets of $[n]$, does there exists a partition of $[n]$ by elements C_j ?

(This problem is NP-complete = NP and NP-hard)

recasting basis pursuit to a linear program

Basis pursuit

$$x^\star \hookrightarrow \text{minimize }_{t\in\mathbb{R}^p} \|t\|_1 \text{ subject to } \Gamma t = \Gamma x.$$

is equivalent to the

linear program

$$\left((\mathbf{z}^+)^\star, (\mathbf{z}^-)^\star\right) \hookrightarrow \text{minimize }_{\mathbf{z}^+,\mathbf{z}^-\in\mathbb{R}^p} \sum_{j=1}^{N} (z_j^+ + z_j^-)$$

$$\text{subject to } \left[\Gamma \middle| -\Gamma\right] \begin{bmatrix} \mathbf{z}^+ \\ \mathbf{z}^- \end{bmatrix} = \Gamma x, \begin{bmatrix} \mathbf{z}^+ \\ \mathbf{z}^- \end{bmatrix} \geq 0.$$

$$x^\star = (\mathbf{z}^+)^\star - (\mathbf{z}^-)^\star$$

Definition

A centrally symmetric polytope $P \subset \mathbb{R}^n$ is said s-neighborly if every set of s vertices, containing no antipodal pair, is the set of all vertices of some faces of P.

Example : ΓB_1^p is s-neighborly when : $\forall I \subset [p], |I| \leq s, (\epsilon_i)_{i \in I} \in \{\pm 1\}^I$,

$$\mathrm{aff}(\{\epsilon_i X_i : i \in I\}) \cap \mathrm{conv}(\{\theta_j X_j, j \notin I, \theta_j \in \{\pm 1\}\}) = \emptyset$$

1. Paley-Zygmund : if $\|Z\|_{2+\epsilon} \leq \kappa \|Z\|_2$,

$$P\big[|Z| \geq (1/2)\|Z\|_2\big] \geq \left[\frac{3\|Z\|_2^2}{4\|Z\|_{2+\epsilon}^2}\right]^{\frac{2+\epsilon}{\epsilon}} \geq \left[\frac{3}{4\kappa}\right]^{\frac{2+\epsilon}{\epsilon}}.$$

2. Einmahl-Mason : if $Z \geq 0$ then for $t > 0$,

$$P[Z \leq \mathbb{E}Z - t\|Z\|_2] \leq \exp(-ct^2).$$

So if $\|Z\|_2 \leq \kappa \|Z\|_1$,

$$P[|Z| \geq (1-t)\|Z\|_2] \geq 1 - \exp(-ct^2).$$

Study the small ball probability function

$$\phi(\epsilon) = P[\|X\|_2 \leq \epsilon] \text{ when } \epsilon \to 0.$$

[Kashin, 1975], [Alon, Goldreich, Hastad, Peralta, 1992], [Devore, 2007], [Nelson, Temlyakov, 2010] $n \gtrsim s^2$

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Still far from the number of mesurements that can be obtained by random matrices : $n \gtrsim s \log(ep/s)$.

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- more measurements : $n \geq 2s$ (for ℓ_0) and $n \gtrsim s\log(ep/s)$ (for ℓ_1).

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- more measurements : $n \geq 2s$ (for ℓ_0) and $n \gtrsim s \log(ep/s)$ (for ℓ_1).

- deterministic measurements for ℓ_0 (the first $2s$ discrete Fourier measurements) to random measurements for ℓ_1.

Proof 1/4

We prove : for any $t \in \sqrt{s}B_1^p \cap S_2^{p-1}$,

$$\|\Gamma t\|_2^2 = \frac{1}{n}\sum_{i=1}^{n}\left\langle X_i, t\right\rangle^2 \geq c_0 > 0.$$

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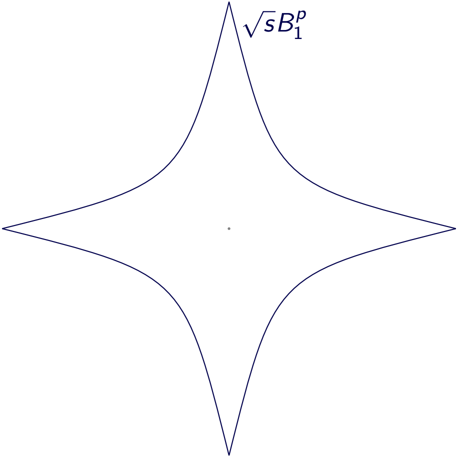

Proof 1/4

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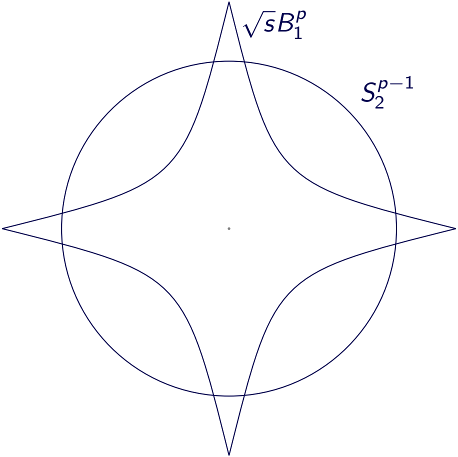

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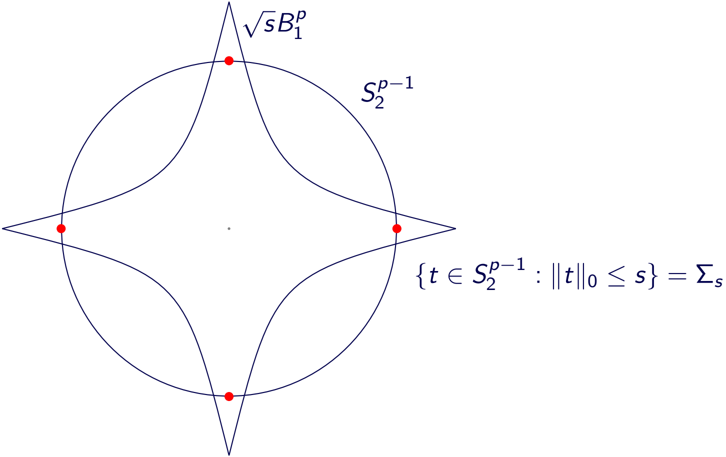

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[Mendelson, Koltchinskii] under moment assumptions.

Proof 3/4 – s-sparse vectors to $\sqrt{s}B_1^p \cap S_2^{p-1}$ via Maurey's method

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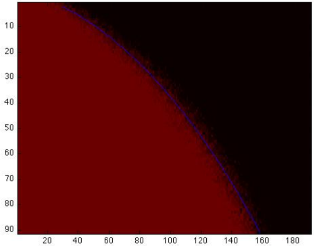
The uniform control $\max_{1\leq j\leq p} \|\Gamma e_j\|_2 \leq c_0$ costs $\log(p)$ moments.

phase transition diagram for Gaussian measurements.



For every (n, s),
n : number of measurements
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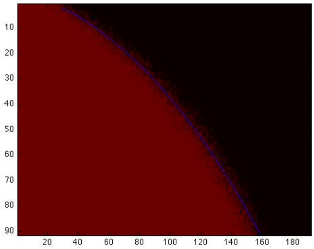

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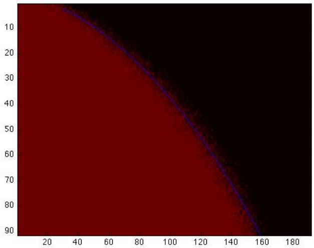

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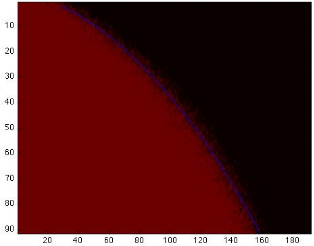

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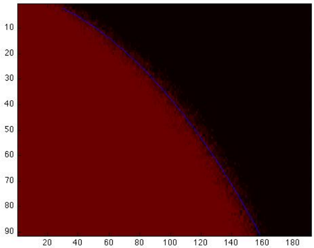

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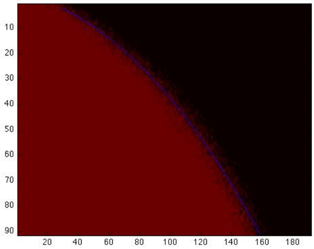

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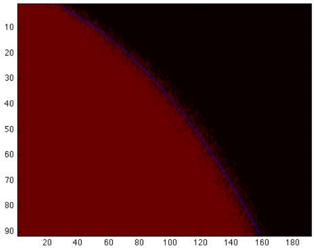

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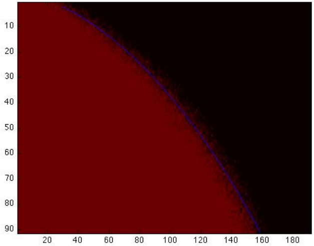
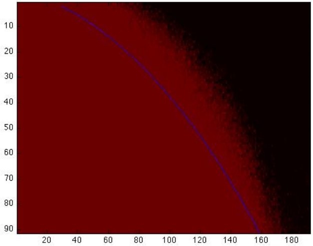
Theoretical phase transition $n \sim s \log(ep/s)$.

phase transition diagram for Gaussian and Cauchy measurements.

Gaussian measurements



Cauchy measurements



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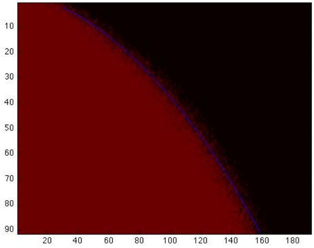

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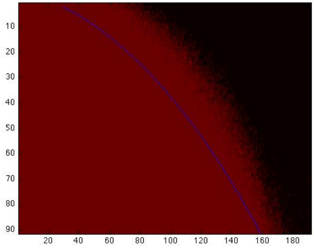

$\log(ep/s)$ moments may be necessary (?)

Smallest singular value of a random matrix

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$\Rightarrow$ Lower bound on the smallest singular value has nothing to do with concentration (true for Cauchy matrices).