From optimal stopping to stochastic optimization

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Outline

Framework

Consider the optimal stopping problem with time$-t$ value

$$U_t = \text{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}[Z_\tau | \mathcal{F}_t]$$

- The non–negative process Z is càdlàg and adapted to the natural filtration $\mathcal{F}$ of $d-$dimensional Brownian motion. Assume $\mathbb{E}\left[\sup_t Z_t^2\right] < \infty$.
- The set $\mathcal{T}_t$ is the set of all $\mathcal{F}-$ stopping times with values in $[t, T]$.
- A typical example is the pricing of an American option with discounted payoff Z.

Dual approach (1)

The *Snell envelope* process $(U_t)_{0 \le t \le T}$ admits a Doob–Meyer decomposition

$$U_t = U_0 + M_t^\star - A_t^\star.$$

[Rogers, 2002]: $$U_0 = \inf_{M \in H_0^1} \mathbb{E}\left[\sup_{0 \le t \le T} (Z_t - M_t)\right] = \mathbb{E}\left[\sup_{0 \le t \le T} (Z_t - M_t^\star)\right]$$

- This problem admits more than a single solution.
- For any stopping time τ smaller than the largest optimal strategy,

$$U_0 = \inf_{M \in H_0^1} \mathbb{E}\left[\sup_{\tau \le t \le T} (Z_t - M_t)\right] = \mathbb{E}\left[\sup_{\tau \le t \le T} (Z_t - M_t^\star)\right].$$

Dual approach (2)

- Some of the martingales M attaining the infimum are surely optimal

$$U_0 = \sup_{0 \leq t \leq T} (Z_t - M_t) \quad a.s.$$

- From [Schoenmakers et al., 2013], any martingale satisfying

$$\mathrm{Var}\left(\sup_{0 \leq t \leq T} (Z_t - M_t) \right) = 0$$

  is surely optimal.
- From [Jamshidian, 2007], for any optimal stopping time τ and any surely optimal martingale M,

$$(M_{t \wedge \tau})_t = (M^\star_{t \wedge \tau})_t.$$

Dual approach (3)

With our square integrability assumption, we can rewrite the minimization problem as

$$U_0 = \inf_{\substack{X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \\ \text{s.t. } \mathbb{E}[X] = 0}} \mathbb{E}\left[\sup_{0 \le t \le T} (Z_t - \mathbb{E}[X|\mathcal{F}_t])\right].$$

How to approximate $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ by a finite dimensional vector space in which conditional expectations are tractable in a closed form?

Truncated Wiener chaos expansion ($d = 1$)

Let H_i the $i - th$ Hermite polynomial.
Take a regular grid $0 = t_0 < t_1 < \cdots < t_n = T$ and $G_i = \frac{B_{t_i} - B_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}$.
Define the truncated Wiener chaos space of order p

$$\mathcal{H}_p = \operatorname{span}\left\{ \prod_{i=1}^{n} H_{\alpha_i}(G_i) \ : \ \alpha \in \mathbb{N}^n, \ \|\alpha\|_1 = p \right\}$$

For $F \in L^2(\Omega, \mathcal{F}_T)$, we introduce the truncated chaos expansion of order p

$$C_{p,n}(F) = \sum_{\alpha \in A_{p,n}} \lambda_\alpha \prod_{i \geq 1} H_{\alpha_i}(G_i) = \sum_{\alpha \in A_{p,n}} \lambda_\alpha \widehat{H}_\alpha(G_1, \ldots, G_n)$$

where $A_{p,n} = \{\alpha \in \mathbb{N}^n \ : \ \|\alpha\|_1 \leq p\}$ with $\|\alpha\|_1 = \sum_{i \geq 0} \alpha_i$.

Key property of the truncated Wiener chaos expansion

For $k \le n$,

$$\mathbb{E}[C_{p,n}(F)|\mathcal{F}_{t_k}] = \sum_{\alpha \in A^k_{p,n}} \lambda_\alpha \, \widehat{H}_\alpha(G_1, \dots, G_n)$$

with $A^k_{p,n} = \{\alpha \in \mathbb{N}^n \ : \ \|\alpha\|_1 \le p, \ \alpha_\ell = 0 \ \forall \ell > k\}$.

"Computing $\mathbb{E}[\cdot|\mathcal{F}_{t_k}]$" $\Leftrightarrow$ "Dropping all non $\mathcal{F}_{t_k}$ – measurable terms"

Extension to the multi–dimensional case

The truncated chaos expansion of order p of $F \in L^2(\Omega, \mathcal{F}_T)$ is given by

$$C_{p,n}(F) = \sum_{\alpha \in A_{p,n}^{\otimes d}} \lambda_\alpha \widehat{H}_\alpha^{\otimes d}(G_1, \dots, G_n) = C_{p,n}(\lambda)$$

where

$$\widehat{H}_\alpha^{\otimes d}(G_1, \dots, G_n) = \prod_{j=1}^{d} \widehat{H}_{\alpha_j}(G_1^j, \dots, G_n^j) \quad \forall \alpha \in (\mathbb{N}^n)^d,$$

$$A_{p,n}^{\otimes d} = \left\{ \alpha \in (\mathbb{N}^n)^d \ : \|\alpha\|_1 \le p \right\}.$$

Return to the optimal stopping problem

We approximate the original problem

$$\inf_{\substack{X \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \\ \text{s.t. } \mathbb{E}[X]=0}} \mathbb{E}\left[\sup_{0 \le t \le T} (Z_t - \mathbb{E}[X|\mathcal{F}_t])\right]$$

by

$$\inf_{\substack{\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}} \\ \text{s.t. } \lambda_0 = 0}} V_{p,n}(\lambda) \tag{1}$$

with

$$V_{p,n}(\lambda) = \mathbb{E}\left[\max_{0 \le k \le n} (Z_{t_k} - \mathbb{E}[C_{p,n}(\lambda)|\mathcal{F}_{t_k}])\right].$$

Properties of the minimization problem (1)

Proposition 1

The minimization problem (1) has at least one solution.

- The function $V_{p,n}$ is clearly convex (maximum of affine functions).
- Not strongly convex but,

$$V_{p,n}(\lambda) \geq \frac{|\lambda|}{2} \inf_{\mu \in \mathbb{R}^{A_{p,n}^{\otimes d}}, |\mu|=1} \mathbb{E}\left[|C_{p,n}(\mu)|\right].$$

Properties of the minimization problem (2)

$$\mathcal{I}(\lambda, Z, G) = \{0 \le k \le n \;:\; \text{the pathwise maximum is attained at time } k\}.$$

Proposition 2

Let $p \geq 1$. Assume that

$$\forall 1 \le r \le k \le n,\ \forall F\ \mathcal{F}_{t_k} - measurable,\ F \in \mathcal{C}_{p-1,n},\ F \ne 0,$$
$$\exists\ 1 \le q \le d \text{ s.t. } \mathbb{P}\left(\forall t \in]t_{r-1}, t_r],\ D_t^q Z_{t_k} + F = 0 \;\middle|\; Z_{t_k} > 0\right) = 0.$$

Then, the function $V_{p,n}$ is differentiable at all points $\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}}$ with no zero component and its gradient $\nabla V_{p,n}$ is given by

$$\nabla V_{p,n}(\lambda) = \mathbb{E}\left[\mathbb{E}\left[\widehat{H}^{\otimes d}(G_1, \dots, G_n) \;\middle|\; \mathcal{F}_{t_i}\right]_{|i=\mathcal{I}(\lambda,Z,G)}\right].$$

Properties of the minimization problem (3)

- Differentiability is ensured as soon as $\mathcal{I}(\lambda, Z, G)$ is a.s. reduced to a unique element: purpose of the blue condition.
- Let $\lambda^\sharp_{p,n}$ be a solution, $V_{p,n}(\lambda^\sharp_{p,n}) = \inf_\lambda V_{p,n}(\lambda)$. Then,

$$\nabla V_{p,n}(\lambda^\sharp_{p,n}) = 0.$$

Convergence to the true solution

Proposition 3

The solution of the minimization problem (1), $V_{p,n}(\lambda^\sharp_{p,n})$, converges to the optimal stopping value U_0 when both p and n go to infinity and moreover

$$0 \le V_{p,n}(\lambda^\sharp_{p,n}) - U_0 \le 2 \left\| M^\star_T - C_{p,n}(M^\star_T) \right\|_2 .$$

Practically solving the optimization problem (1)

We approximate the solution of

$$V_{p,n}(\lambda^{\sharp}_{p,n}) = \inf_{\lambda \in A_{p,n}^{\otimes d}} V_{p,n}(\lambda) = \inf_{\lambda \in A_{p,n}^{\otimes d}} \mathbb{E}\left[\max_{0\leq k\leq n}\left(Z_{t_k} - \mathbb{E}[C_{p,n}(\lambda)|\mathcal{F}_{t_k}]\right)\right]$$

by introducing the well–known *Sample Average Approximation* (see [Rubinstein and Shapiro, 1993]) of $V_{p,n}$ defined by

$$V^m_{p,n}(\lambda) = \frac{1}{m}\sum_{i=1}^{m} \max_{0\leq k\leq n}\left(Z^{(i)}_{t_k} - \mathbb{E}[C^{(i)}_{p,n}(\lambda)|\mathcal{F}_{t_k}]\right).$$

Note that the conditional expectation boils down to truncating the chaos expansion and hence is tractable in a closed form.

Practically solving the optimization problem (2)

For large enough m, $V^m_{p,N}$ is convex, a.s. differentiable and tends to infinity at infinity. Then, there exits $\lambda^m_{p,n}$ such that

$$V^m_{p,n}(\lambda^m_{p,n}) = \inf_{\lambda \in \mathbb{R}^{A_{p,n}^{\otimes d}}} V^m_{p,n}(\lambda).$$

Proposition 4

*$V^m_{p,n}(\lambda^m_{p,n})$ converges a.s. to $V_{p,n}(\lambda^\sharp_{p,N})$ when $m \to \infty$.
The distance from $\lambda^m_{p,n}$ to the set of minimizers of $V_{p,n}$ converges to zero as m goes to infinity.*

Practically solving the optimization problem (3)

Write $M_k(\lambda) = \mathbb{E}[C_{p,n}(\lambda)|\mathcal{F}_{t_k}]$ for $0 \le k \le n$.

Proposition 5

Assume $\lambda^{\sharp}_{p,n}$ is unique. Then,

$$\frac{1}{m}\sum_{i=1}^{m}\left(\max_{0\le k\le n} Z^{(i)}_{t_k} - M^{(i)}_k(\lambda^m_{p,n})\right)^2 - V^m_{p,n}(\lambda^m_{p,n})^2$$

is a convergent estimator of $\mathrm{Var}(\max_{k\le 0\le n} Z_{t_k} - M_k(\lambda^{\sharp}_{p,n}))$ *and moreover, if* $\lambda^m_{p,n}$ *is bounded,*

$$\lim_{m\to\infty} m \,\mathrm{Var}\left(V^m_{p,n}(\lambda^m_{p,n})\right) = \mathrm{Var}(\max_{k\le 0\le n} Z_{t_k} - M_k(\lambda^{\sharp}_{p,n})).$$

The algorithm: bespoke martingales

Define the first time the option goes in the money by

$$\tau_0 = \inf\{k \geq 0 \ : \ Z_{t_k} > 0\} \wedge n.$$

Consider martingales only starting once the option has been in the money

$$N_k(\lambda) = M_k(\lambda) - M_{k \wedge \tau_0}(\lambda).$$

In the dual price, "$\max_{0 \leq k \leq n}$" can be shrunk to "$\max_{\tau_0 \leq k \leq n}$".
Using Doob's stopping theorem, we have

$$\mathbb{E}\left[\max_{\tau_0 \leq k \leq n} (Z_{t_k} - M_k(\lambda))\right] = \mathbb{E}\left[\max_{\tau_0 \leq k \leq n} (Z_{t_k} - (M_k(\lambda) - M_{\tau_0}(\lambda)))\right]$$

The martingales $M(\lambda)$ or $N(\lambda)$ lead to the same minimum value.
The set of martingales N^λ is far more efficient from a practical point of view.

The algorithm: a gradient descent with line search

$x_0 \leftarrow 0$, $k \leftarrow 0$, $\gamma \leftarrow 1$, $d_0 \leftarrow 0$, $v_0 \leftarrow \infty$;
while *True* **do**
  Compute $v_{k+1/2} \leftarrow V^m_{p,n}(x_k - \gamma\alpha_k d_k)$;
  if $v_{k+1/2} < v_k$ **then**
    $x_{k+1} \leftarrow x_k - \gamma\alpha_k d_k$;
    $v_{k+1} \leftarrow v_{k+1/2}$;
    $d_{k+1} \leftarrow \nabla V^m_{p,n}(x_{k+1})$;
    if $\frac{|v_{k+1} - v_k|}{v_k} \leq \varepsilon$ **then return**;
  else
    $\gamma \leftarrow \gamma/2$;
  end
end

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    $x_{k+1} \leftarrow x_k - \gamma\alpha_k d_k$;
    $v_{k+1} \leftarrow v_{k+1/2}$;
    $d_{k+1} \leftarrow \nabla V^m_{p,n}(x_{k+1})$;
    if $\frac{|v_{k+1}-v_k|}{v_k} \leq \varepsilon$ **then return**;
  else
    $\gamma \leftarrow \gamma/2$;
  end
end

Take $\alpha_\ell = \dfrac{V^m_{p,n}(x_\ell) - \mathbb{E}[Z_T]}{\left\|\nabla \tilde{V}^m_{p,n}(x_\ell)\right\|^2}$, see [Polyak, 1987].

Some remarks on the algorithm

- Given the expression of $V^m_{p,n}$, both the value function and its gradient are computed at the same time without extra cost.

$$V_{p,n}(\lambda) = \mathbb{E}\left[\max_{\tau_0 \le k \le n} \left(Z_{t_k} - \mathbb{E}[\lambda \cdot H^{\otimes d}(G_1, \cdots, G_n) | \mathcal{F}_{t_k}]\right)\right],$$
$$= \mathbb{E}[Z_{t_{\mathcal{I}(\lambda,Z,G)}}] - \lambda \cdot \nabla \tilde{V}_{p,n}(\lambda).$$

- Checking the admissibility of a step γ costs as much as updating x_k.
- The algorithm is *almost* embarrassingly parallel:
  - Few iterations of the gradient descent are required (≈ 10).
  - Each iteration is fully parallel: each process treats its bunch of paths.
  - No demanding centralized computations
  - Very little communication: a few broadcasts only.

Parallel implementation

In parallel *Generate* $(G^{(1)}, Z^{(1)}), \dots, (G^{(m)}, Z^{(m)})$ m $x_0 \leftarrow 0 \in \mathbb{R}^{A_{p,n}^{\otimes d}}$;
while *True* **do**
  Broadcast x_ℓ, d_ℓ, γ, α_ℓ;
  In parallel *Compute* $\max_{\tau_0 \le k \le n}(Z_{t_k}^{(i)} - N_k^{(i)}(x_\ell - \gamma\alpha_\ell d_\ell))$;
  Make a reduction of *the above contributions to obtain* $V_{p,n}^m(x_{\ell+1/2})$ *and* $\nabla V_{p,n}^m(x_{\ell+1/2})$;
  $v_{\ell+1/2} \leftarrow V_{p,n}^m(x_\ell - \gamma\alpha_\ell d_\ell)$;
  if $v_{\ell+1/2} < v_\ell$ **then**
    $x_{\ell+1} \leftarrow x_\ell - \gamma\alpha_\ell d_\ell$;
    $v_{\ell+1} \leftarrow v_{\ell+1/2}$; $d_{\ell+1} \leftarrow \nabla V_{p,n}^m(x_{\ell+1})$;
    if $\frac{|v_{\ell+1} - v_\ell|}{v_\ell} \le \varepsilon$ **then return**;
  else
    $\gamma \leftarrow \gamma/2$;
  end
end

Basket option in the BS model

p	n	S_0	price	Stdev	time (sec.)	reference price
2	3	100	2.27	0.029	0.17	2.17
3	3	100	2.23	0.025	0.9	2.17
2	3	110	0.56	0.014	0.07	0.55
3	3	110	0.53	0.012	0.048	0.55
2	6	100	2.62	0.021	0.91	2.43
3	6	100	2.42	0.021	14	2.43
2	6	110	0.61	0.012	0.33	0.61
3	6	110	0.55	0.008	10	0.61

TAB.: Prices for the put basket option with parameters $T = 3$, $r = 0.05$, $K = 100$, $\rho = 0$, $\sigma^j = 0.2$, $\delta^j = 0$, $d = 5$, $\omega^j = 1/d$, $m = 20,000$.

Scalability of the parallel algorithm

The tests were run on a BullX DLC supercomputer containing 3204 cores.

#processes	time (sec.)	efficiency
1	4365	1
2	2481	0.99
4	1362	0.90
16	282	0.84
32	272	0.75
64	87	0.78
128	52	0.73
256	34	0.69
512	10.7	0.59

TAB.: Scalability of the parallel algorithm on the $40-$dimensional geometric put option described above with $T = 1$, $r = 0.0488$, $K = 100$, $\sigma^j = 0.3$, $\rho = 0.1$, $\delta^j = 0$, $n = 9$, $p = 2$, $m = 200,000$.

Conclusion

- Purely optimization approach. No need of an optimal strategy.
- The problem is in large dimension but convex.
- *Almost* embarrassingly parallel and scales very well.
- Can deal with path dependent options

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