The charged polymer model

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(based on joint works with Berger, Caravenna, den Hollander, Pétrélis)

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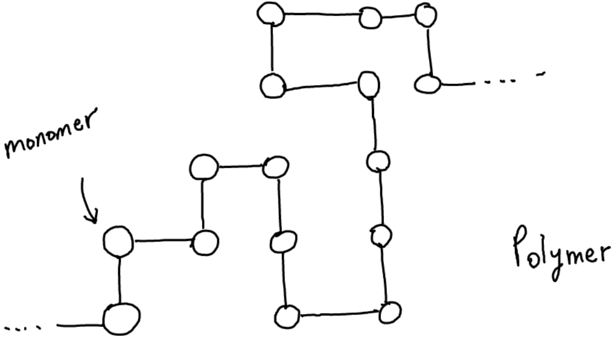
August 30, 2016

Motivation



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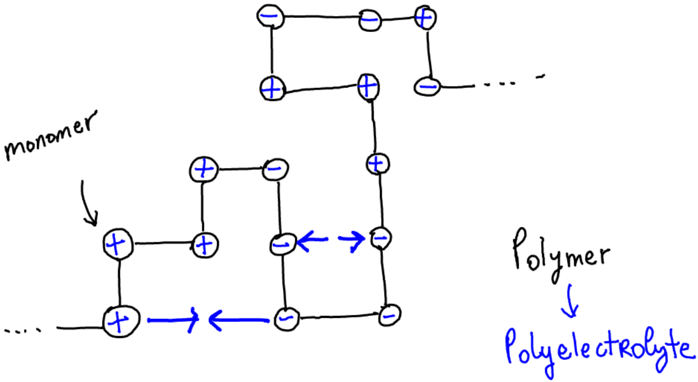

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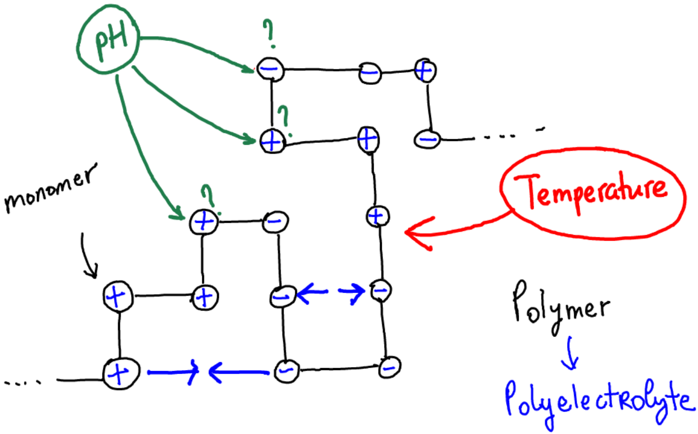

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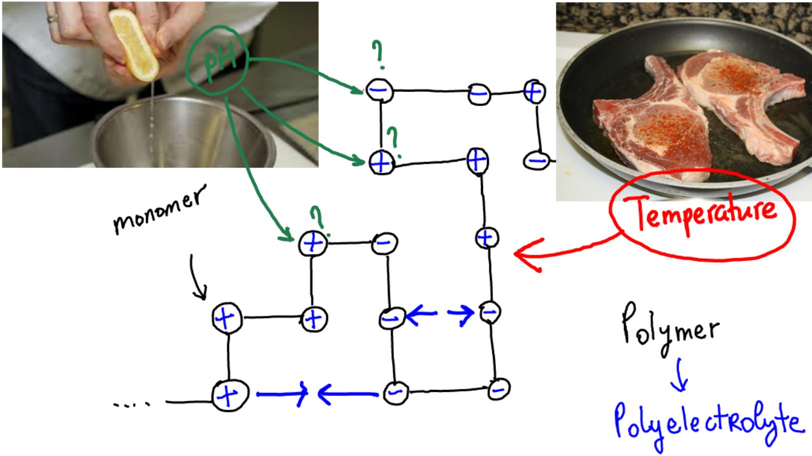
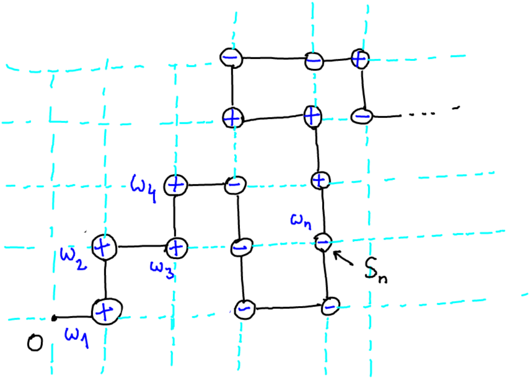


Model

- polymer chain: $S = (S_n)_{n\geq 0}$ SRW on $\mathbb{Z}^d$ ($\mathbf{P}$)
- charges: $\omega = (\omega_n)_{n\geq 0}$ iid, ± 1 with prob. $1/2$ ($\mathbb{P}$),



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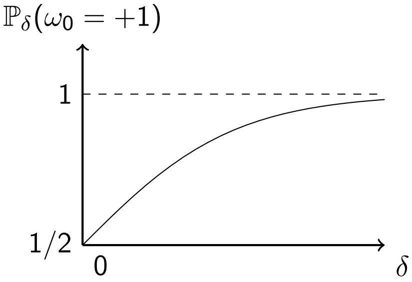
- tilting parameter $\delta \geq 0$

$$\frac{\mathrm{d}\mathbb{P}_\delta}{\mathrm{d}\mathbb{P}} = \frac{e^{\delta\omega_0}}{M(\delta)}$$



Annealed polymer measure

- On-site interaction:

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- Annealed polymer measure:

$$\frac{\mathrm{d}\mathbf{P}_n^{\beta,\delta}}{\mathrm{d}(\mathbf{P} \times \mathbb{P}_\delta)}(\omega, S) = \frac{\exp(-\beta H_n)}{\mathbb{Z}_n^{\beta,\delta}},$$

where β is the **inverse temperature**, and $\mathbb{Z}_n^{\beta,\delta}$ is the **annealed partition function.**

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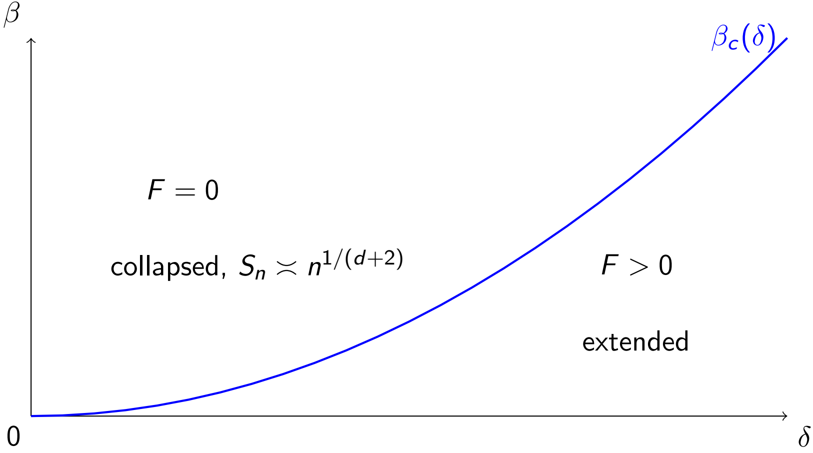
- $d = 1$: the limit **exists** and has a **spectral representation**;
- $d \geq 2$: **open**, but we can replace lim by lim sup...

Annealed phase transition



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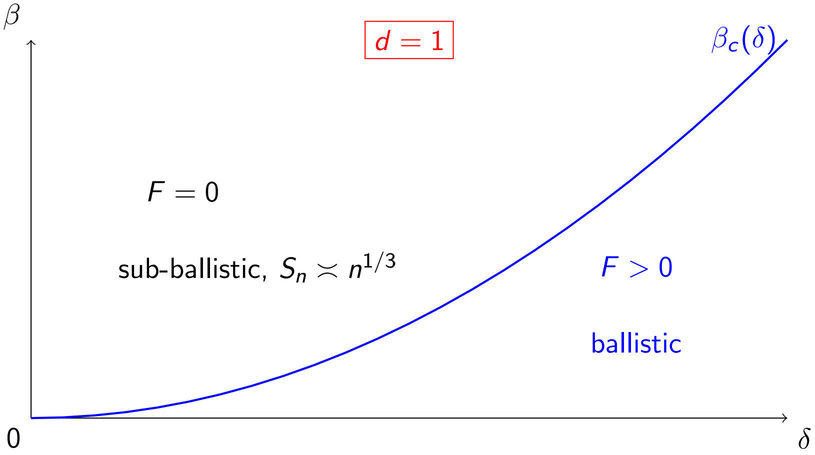

Spectral representation

- Key formula:

$$\mathbb{Z}_n^{\beta,\delta} = \mathbf{E}\left(\prod_{x\in\mathbb{Z}^d} g_{\delta,\beta}(\ell_n(x))\right), \quad \ell_n(x) = \sum_{i=1}^{n} \mathbf{1}\{S_i = x\};$$

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Spectral representation

$$F^{ann}(\beta,\delta) = \inf\{\mu \geq 0 \colon \mathrm{spr}(A_{\mu,\beta,\delta}) \leq 1\}$$

Weak-coupling limits

As $\delta \searrow 0$,

$$\beta_c(\delta) = \frac{1}{2}\delta^2 - [1 + o(1)] \begin{cases} c_1\delta^{8/3} & d = 1 \\ c_2\delta^4|\log\delta| & d = 2 \ \textbf{(conj)} \\ c_d\delta^4 & d \geq 3 \ \textbf{(conj)} \end{cases}$$

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- Related to the self intersection local time of the SRW,
- The constant c_1 is universal and the principal eigenvalue of a Sturm-Liouville operator

Quenched model

Quenched polymer measure:

$$\frac{\mathrm{d}\mathbf{P}^{\omega}_{n,\beta}}{\mathrm{d}\mathbf{P}}(S) = \frac{\exp(-\beta H_n)}{Z^{\omega}_{n,\beta}},$$

where $Z^{\omega}_{n,\beta}$ is the **quenched partition function.**

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Many open questions:

- Existence, self-averaging of the free energy?
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Some answers:

- If the average charge is non-zero, the number of visited sites is linear in n;
- Ballistic behaviour in $d = 1$ for large average charges and low temperature.

References

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- den Hollander: Random Polymers, 2009.
- Caravenna, den Hollander, Pétrélis, P': Annealed scaling for a charged polymer. *Mathematical Physics, Analysis and Geometry*, 2016.
- Berger, den Hollander, P': Annealed scaling for a charged polymer in dimensions two and higher. *[In progress].*