Perfect sampling for closed queueing networks

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1 Introduction
- Perfect sampling
- Closed queueing network

2 Efficient perfect sampling

3 Conclusion

Sample the stationary distribution

- Ergodic Markov chain $(X_n)_{n \in Z}$ on $\mathcal{S}$
- Stationary distribution π (unknown)

Sample the stationary distribution

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Question

How do we simulate a random object with distribution π ?

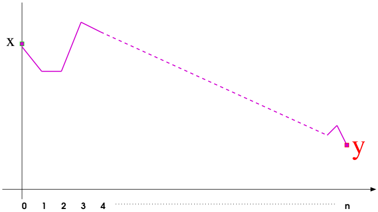
Markov Chain Monte Carlo (MCMC)

Markov chain convergence theorem

For all initial distribution $n \to +\infty$ $X_n \sim \pi$

- $(U_n)_{n \in Z}$ an i.i.d sequence of random variables

$$\begin{cases} X_0 = x \in \mathcal{S} \\ X_{n+1} = F_{U_n}(X_n) \end{cases}$$



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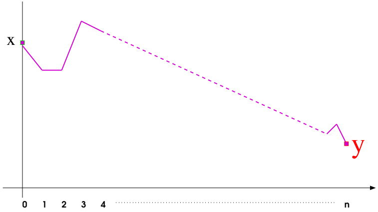
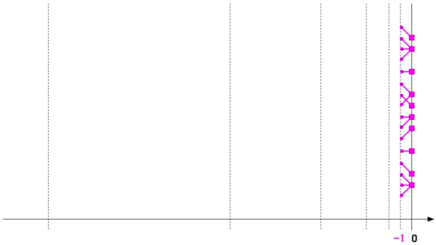
- How to choose n ?



Perfect sampling algorithm

- Ergodic Markov chain $(X_n)_{n \in Z}$ on $\mathcal{S}$, stationary distribution π
- Perfect sampling algorithm [Propp Willson, 1996]
  - Produces $y \sim \pi$
  - Detects n
  - Uses coupling from the past



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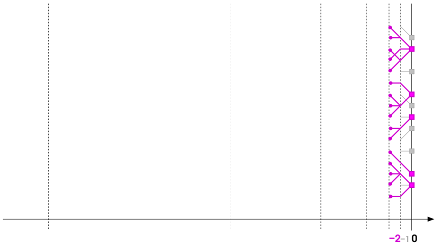

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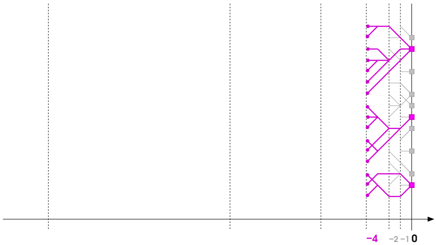

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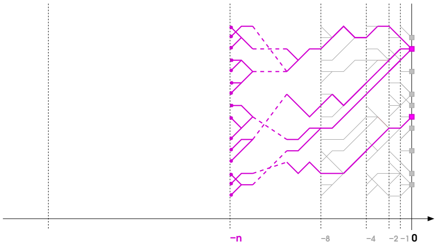

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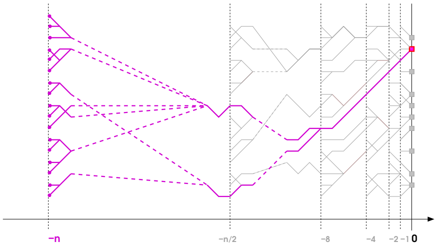
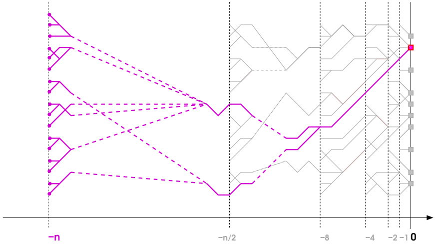


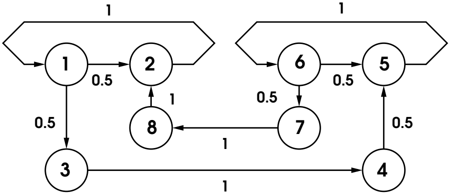
Perfect sampling algorithm

- Ergodic Markov chain $(X_n)_{n \in Z}$ on $\mathcal{S}$, stationary distribution π
- Perfect sampling algorithm [Propp Willson, 1996]
  - Produces $y \sim \pi$
  - Detects n
  - Uses coupling from the past
- Starts with all states, complexity at least in $O(|\mathcal{S}|)$
- Find stategies (monotone chains, envelope, . . .)



Closed queueing network

- Closed queuing network
  - K queues $./M/1$ (exponential service rate)
  - Queues have finite capacity C_k
  - Blocking policy: Repetitive service - random destination
  - Customers are not allowed leave the network



- Evolution of the network modelised by an ergodic Markov chain

State space

- State space:

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{N}^K \mid \sum_{k=1}^{K} x_k = M, \; \forall k \; 0 \leq x_k \leq C_k\}$$

- Number of states:

$$|\mathcal{S}| \leq \binom{M+K-1}{M} = \binom{M+K-1}{K-1} \text{ in } O\left(\frac{(M+K-1)!}{(K-1)!M!}\right)$$

- For $M >> K$ number of states in $O(M^K)$

Example

- $K = 8, \; C = (3,3,3,3,3,3,3,3), \; M = 4$
- State $\mathbf{x} = (0,2,0,0,1,0,0,1)$ (a possible configuration)
- Number of states: $|\mathcal{S}| = 322$

Transition function

- Let $(i,j) \in \{1,2,\ldots,K\}^2$
- **Transition function**: $t_{i,j} \ : \ \mathcal{S} \rightarrow \mathcal{S}$

$$t_{i,j}(\mathbf{x}) = \begin{cases} \mathbf{x} - e_i + e_j & \text{if } x_i > 0 \text{ and } x_j < C_j, \\ x & \text{otherwise } (x_i = 0 \text{ or } x_j = C_j), \end{cases}$$

where $e_i \in \{0,1\}^K$

$$e_i(k) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

- $S \subseteq \mathcal{S}$, $t(S) := \bigcup_{\mathbf{x} \in S} t(\mathbf{x})$

Markov chain

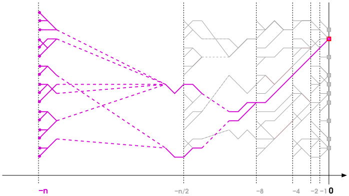
- $(U_n)_{n \in Z} = (i_n, j_n)_{n \in Z}$ an i.i.d sequence of random variables
- The evolution of the system can be described by an ergodic Markov chain:
$$\begin{cases} X_0 \in \mathcal{S} \\ X_{n+1} = t_{U_n}(X_n) \end{cases}$$
- Unique stationary distribution π that is unknown
- GOAL: sample π with the perfect sampling algorithm

Perfect sampling algorithm

Algorithm

1. $n \leftarrow 1$
2. $t \leftarrow t_{U_{-1}}$
3. While $|t(\mathcal{S})| \neq 1$
4. $\quad n \leftarrow 2n$
5. $\quad t \leftarrow t_{U_{-1}} \circ \ldots \circ t_{U_{-n}}$
6. Return $t(\mathcal{S})$

- $t(S) := \bigcup_{\mathbf{x} \in S} t(\mathbf{x})$



- PROBLEM: $|\mathcal{S}|$ in $O(M^K)$
- Find a strategie !

1 Introduction

2 Efficient perfect sampling
- Definitions
- Transition algorithm
- Exact sampling

3 Conclusion

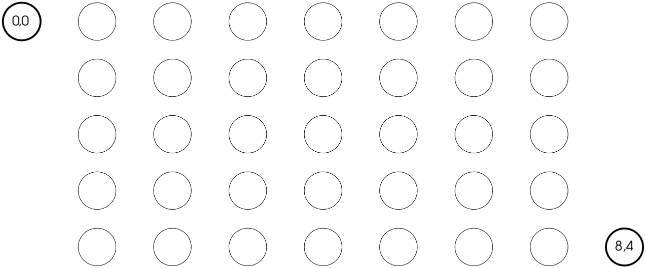
Idea

More structured representation of the state space

- Represent states as paths in a graph
- Realize transitions directly on the graph

Diagram

- State:
  - $x = 02001001$
- Diagram



- $K = 8,\ M = 4,\ C = (3, \ldots, 3)$
- Constraints:
  - $\sum_{k=1}^{8} x_k = 4$
  - $\forall k\ 0 \leq x_k \leq 3$

Diagram

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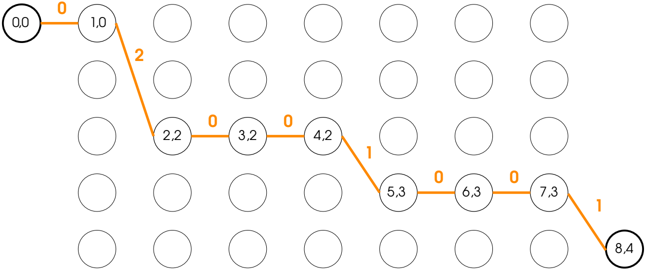

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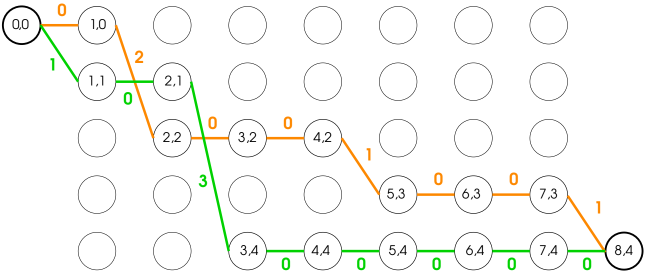
Diagram

- State:
  - $x = 02001001$
  - $y = 10300000$
- Diagram



- $K = 8,\ M = 4,\ C = (3, \ldots, 3)$
- Constraints:
  - $\sum_{k=1}^{8} x_k = 4$
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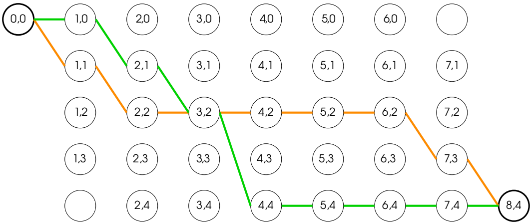
Diagram

- Let $D = (N, A)$ a directed graph:
  - $N = \{0, \dots, K\} \times \{0, \dots, M\}$
  - $g \ : \ \mathcal{S} \to \mathcal{P}(N^2)$
- D is a diagram if $\exists S \subseteq \mathcal{S}$ s.t. $A = g(S) := \bigcup_{s \in S} \{g(s)\}$

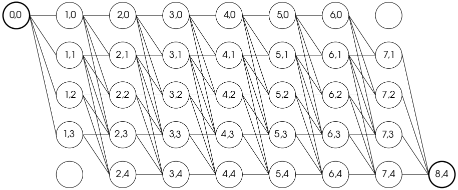


Complete diagram

- $D = (N, A)$ is a complete diagram if $A = g(\mathcal{S})$
- $|A| \leq \frac{K(M+2)(M+1)}{2}$

Example

$K = 8$ queues	$8 + 1$ columns
$M = 4$ customers	$4 + 1$ rows
$C = (3, 3, 3, 3, 3, 3, 3, 3)$	$0 \leq \lvert\text{slopes}\rvert \leq 3$
35 nodes	104 arcs



Why choose diagram representation?

State space vs. complete diagram

$K = 8$ queues	9 columns
$M = 4$ customers	5 rows
$C = (3, 3, 3, 3, 3, 3, 3, 3)$	$0 \leq \lvert\text{slopes}\rvert \leq 3$
$\lvert\mathcal{S}\rvert = 322$ states	104 arcs

Why choose diagram representation?

State space vs. complete diagram

$K = 8$ queues	9 columns
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$C = (3, 3, 3, 3, 3, 3, 3, 3)$	$0 \leq \lvert\text{slopes}\rvert \leq 3$
$\lvert\mathcal{S}\rvert = 322$ states	104 arcs

State space vs. complete diagram

$K = 25$ queues	26 columns
$M = 100$ customers	101 rows
$C = (10, \ldots, 10)$	$0 \leq \lvert\text{slopes}\rvert \leq 10$
$\lvert\mathcal{S}\rvert = 7.9\ 10^{23}$ states	15400 arcs

States to Diagram

- Function ϕ transforms a set of states into its representative diagram.

S

01120000
11000011

Diagram $\phi(S)$

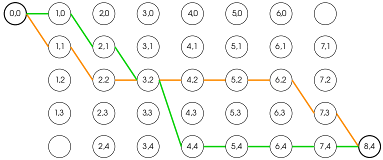
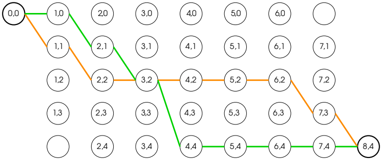


Diagram to states

- Function ψ transforms a diagram into its representative set of states.

Diagram D



$\psi(D)$

01120000
11000011
01100011
11020000

Transition on the diagram

- Let $(i, j) \in R$, **transition function** $T_{i,j}(D)$:

$$T_{i,j}(D) = \phi \circ t_{i,j} \circ \psi(D)$$

Proposition

(i) *If* $S \subseteq \psi(D)$ *then* $t_{i,j}(S) \subseteq \psi(T_{i,j}(D))$

(ii) *If* $|\psi(D)| = 1$ *then* $|\psi \circ T_{i,j}(D)| = 1$

Transition algorithm

$$T_{i,j}(D)$$

Transition function algorithm

Transition $t_{1,3}$ on S

- Chosen policy: Repetitive service - random destination
- Parameters: $K = 8$ queues, $M = 4$ customers, capacity $C = (3, 3, 3, 3, 3, 3, 3, 3)$.
- Transition $t_{1,3}$

$S \subseteq \mathcal{S}$

01100002
01300000
20001001
10300000
20101000
00111100
10210000

Transition $t_{1,3}$ on S

S	$t_{1,3}(\mathbf{x}) = \mathbf{x}$ $x_1 = 0$ OR $x_3 = C_3$
01100002	●
01300000	● ●
20001001	
10300000	●
20101000	
00111100	●
10210000	

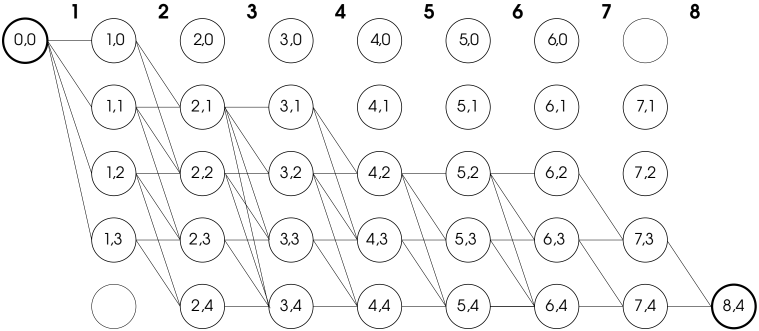
Transition $t_{1,3}$ on S

S	$t_{1,3}(\mathbf{x}) = \mathbf{x}$	$t_{1,3}(x) \neq \mathbf{x}$	
	$x_1 = 0$ OR $x_3 = C_3$	$x_1 > 0$ AND $x_3 < C_3$	
01100002	●		01100002
01300000	● ●		01300000
20001001		●	
10300000	●		10300000
20101000		●	
00111100	●		00111100
10210000		●	

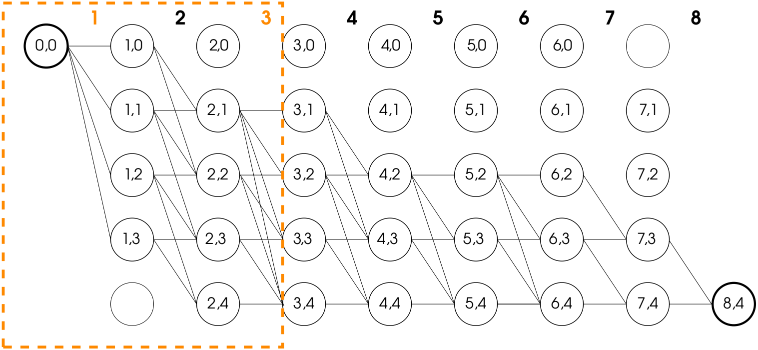
Transition $t_{1,3}$ on S

S	$t_{1,3}(\mathbf{x}) = \mathbf{x}$ $x_1 = 0$ OR $x_3 = C_3$	$t_{1,3}(x) \neq \mathbf{x}$ $x_1 > 0$ AND $x_3 < C_3$	$t_{1,3}(S)$
01100002	•		01100002
01300000	• •		01300000
20001001		•	10101001
10300000	•		10300000
20101000		•	10201000
00111100	•		00111100
10210000		•	00310000

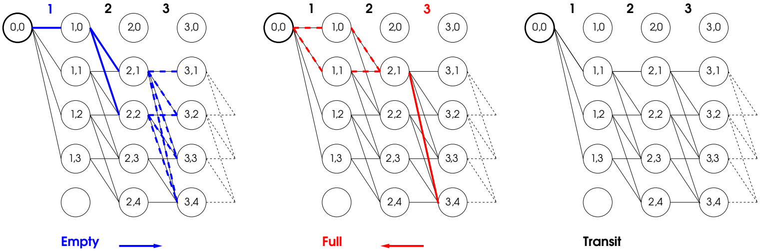
Compute $T_{1,3}(D)$ on D



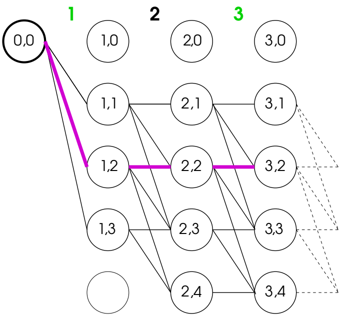
$T_{1,3}(D)$ - What will change:



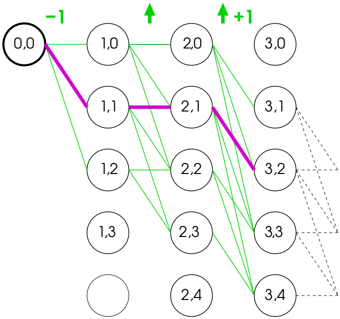
Step 1: Determine subsets



Step 2: Compute $\mathcal{T}\mathrm{ransit}'$

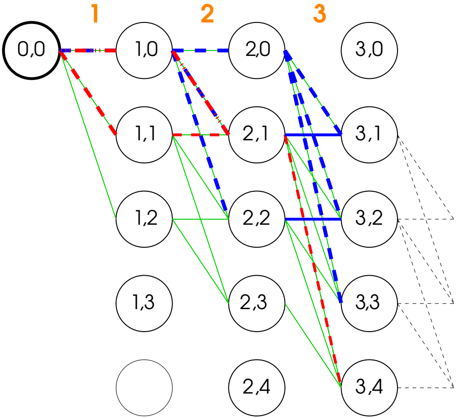


Transit

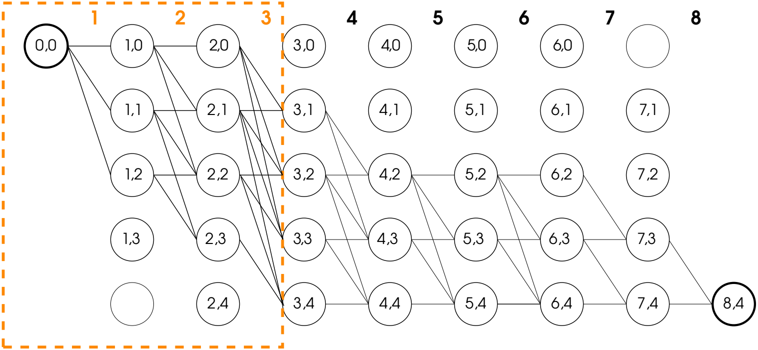


Transit'

Step 3: Compute $\mathcal{E}\mathrm{mpty} \cup \mathcal{F}\mathrm{ull} \cup \mathcal{T}\mathrm{ransit}'$



Step 4: Return $T1,3(D)$



Transition algorithm

- Computation of $T_{i,j}(D)$, with $D = (N, A)$
  1. Determine the subsets $\mathcal{E}\mathrm{mpty}$, $\mathcal{F}\mathrm{ull}$ and $\mathcal{T}\mathrm{ransit}$
  2. Compute $\mathcal{T}\mathrm{ransit}'$
  3. Compute $A' = \mathcal{E}\mathrm{mpty} \cup \mathcal{F}\mathrm{ull} \cup \mathcal{T}\mathrm{ransit}'$
  4. Return $D' = (N, A')$
- The transition algorithm $T_{i,j}$ has a complexity in $O(KM^2)$

Exact sampling on diagram

- $\mathcal{S} = \psi(\mathcal{D})$
- Transitions preserve inclusions:

$$S \subseteq \psi(D) \implies t_{i,j}(S) \subseteq T_{i,j}(\psi(D))$$

- If $|\psi(D)| = 1$ then $|\psi \circ T_{i,j}(D)| = 1$
- (Be proved that) There exists a finite sequence of transitions $T = T_{i_\rho, j_\rho} \circ \cdots \circ T_{i_1, j_1}$ such that $|\psi(T(\mathcal{D}))| = 1$

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- One can use the exact sampling technique on a diagram!

Perfect sampling algorithm

States

$n \leftarrow 1$

$t \leftarrow t_{U_{-1}}$

While $|t(\mathcal{S})| \neq 1$

$\quad n \leftarrow 2n$

$\quad t \leftarrow t_{U_{-1}} \circ \ldots \circ t_{U_{-n}}$

Return $t(\mathcal{S})$

Diagram

$n \leftarrow 1$

$T \leftarrow T_{U_{-1}}$

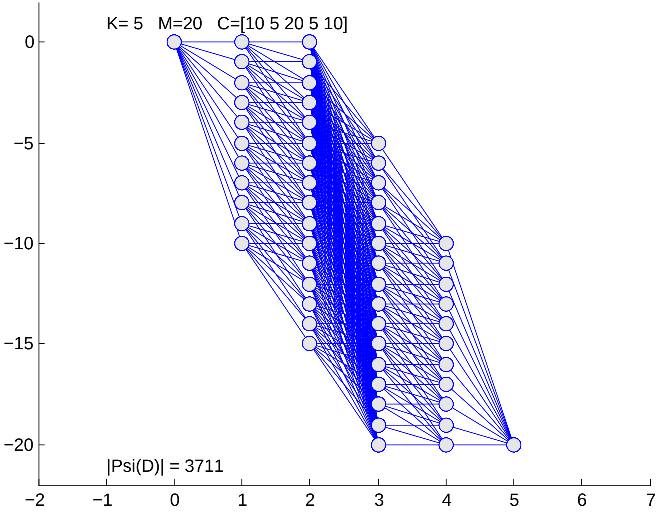
While $|\psi(T(\mathcal{D}))| \neq 1$

$\quad n \leftarrow 2n$

$\quad T \leftarrow T_{U_{-1}} \circ \ldots \circ T_{U_{-n}}$

Return $\psi(T(\mathcal{D}))$

Exact sampling with a diagram



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Conclusion

- More efficient representation in $O(KM)$
- Adapt the method to other classes of queuing networks
  - Multiclass closed queuing networks
  - Queuing networks with synchronisations
- Investigate the coupling time

Thank you, merci

Thank you, merci