Asymptotic behavior of the iteration of a cellular automaton on a probability measure

Journée MAS 2016
Grenoble

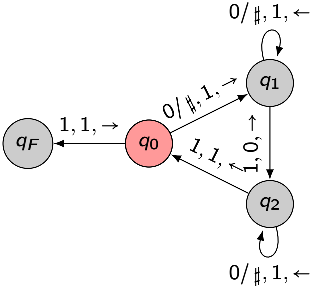
Mathieu Sablik

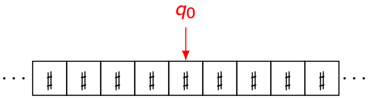
29 août 2016

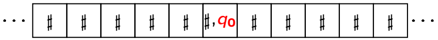
Problematic

Algorithm and probability

An example of model of computation:







Algorithm and probability

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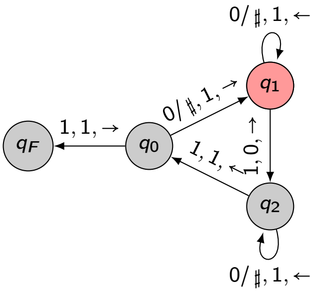

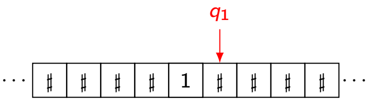

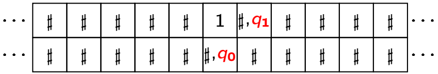

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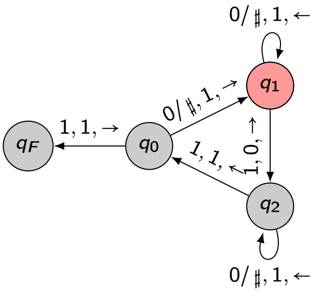

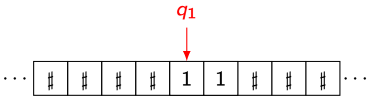

$\cdots$	♯	♯	♯	♯	♯	$1, q_1$	1	♯	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	1	$\sharp, q_1$	♯	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	$\sharp, q_0$	♯	♯	♯	♯	♯	$\cdots$

Algorithm and probability

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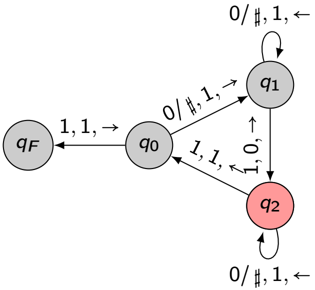

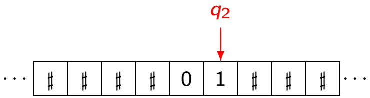

$\cdots$	♯	♯	♯	♯	♯	0	$1, q_2$	♯	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	$1, q_1$	1	♯	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	1	$\sharp, q_1$	♯	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	$\sharp, q_0$	♯	♯	♯	♯	♯	$\cdots$

Algorithm and probability

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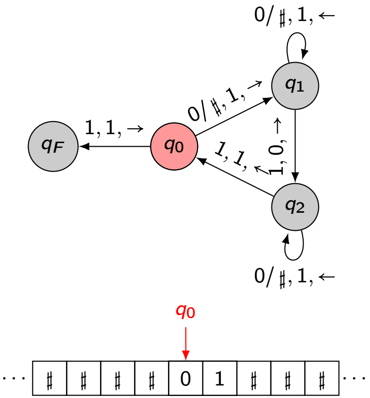

…	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	…
…	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	…

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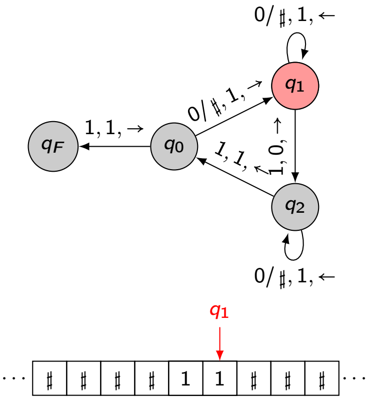
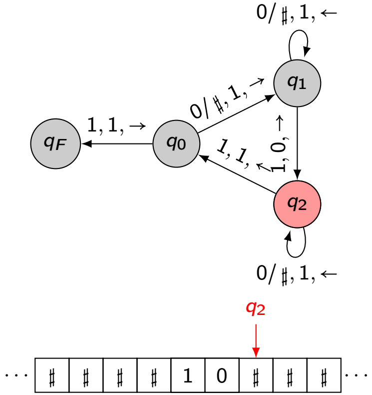


$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

An example of model of computation:



$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp, q_2$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
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$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

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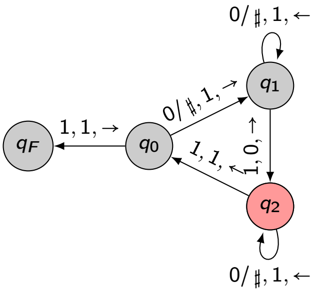

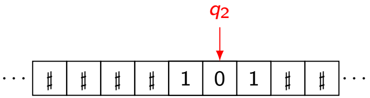

$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$0, q_2$	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp, q_2$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

Algorithm and probability

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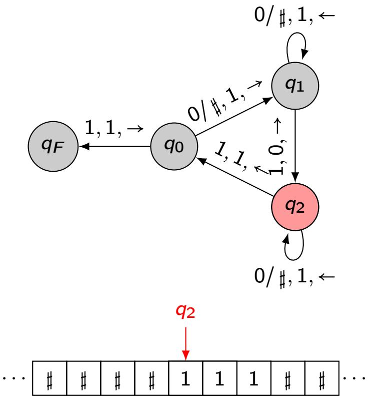

$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_2$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$0, q_2$	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp, q_2$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

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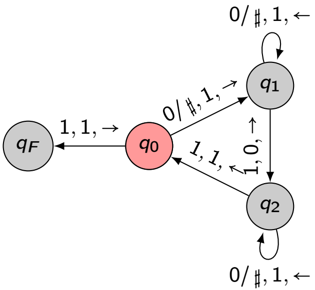

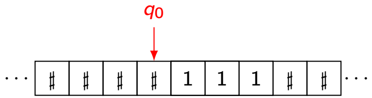

$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	1	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_2$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$0, q_2$	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp, q_2$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

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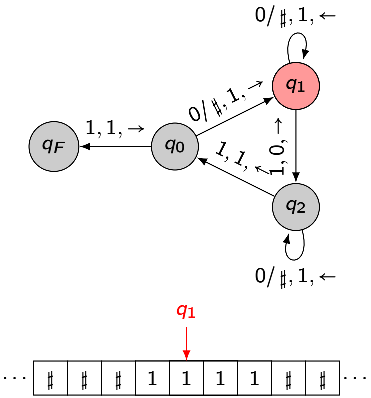

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
$\cdots$	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	$\cdots$
$\cdots$	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	$\cdots$

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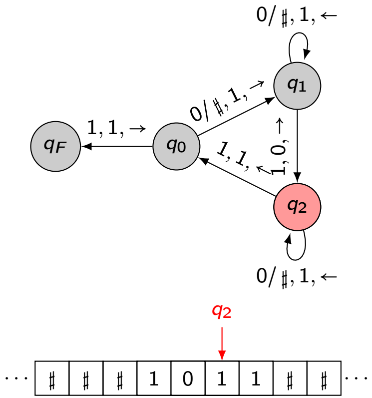

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

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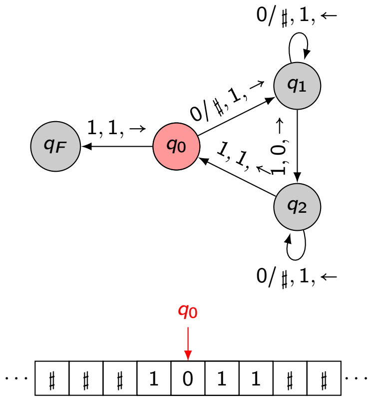

	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	1	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_2$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$0, q_2$	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp, q_2$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

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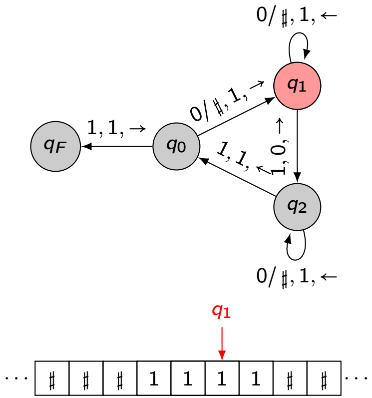

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	1, q_1	1	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$, q_0	1	1	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1, q_2	1	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0, q_2	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp$, q_2	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	1, q_1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0, q_0	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	1, q_2	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1, q_1	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp$, q_1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$, q_0	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯

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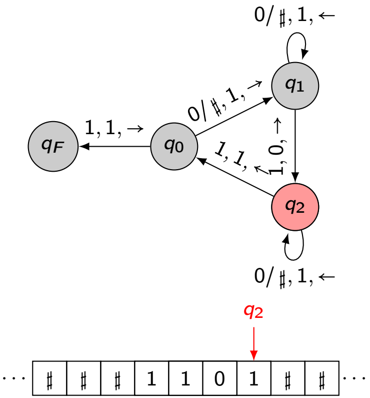
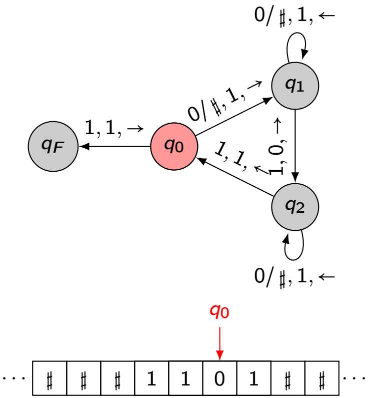


	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	1	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_2$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$0, q_2$	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp, q_2$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

Algorithm and probability

An example of model of computation:



	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	1, q_1	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$, q_0	1	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1, q_2	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0, q_2	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp$, q_2	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	1, q_1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0, q_0	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	1, q_2	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1, q_1	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp$, q_1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$, q_0	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

Algorithm and probability

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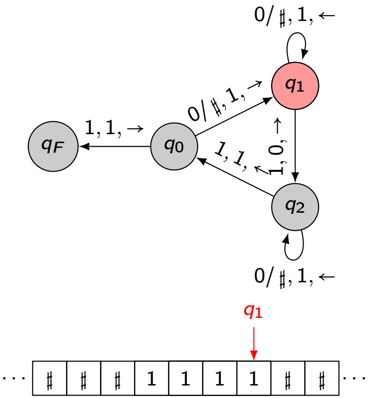

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

Algorithm and probability

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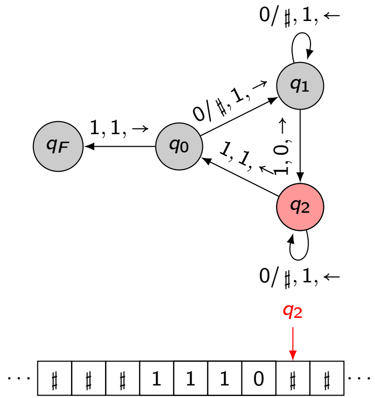
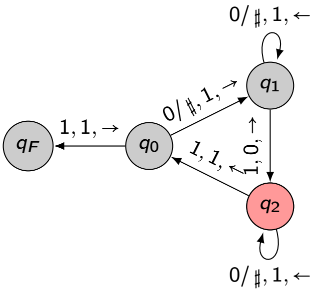


	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	$1,q_1$	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	$\sharp,q_0$	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$1,q_2$	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	$0,q_2$	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	$\sharp,q_2$	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	$1,q_1$	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$0,q_0$	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	$1,q_2$	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$1,q_1$	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	$\sharp,q_1$	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	$\sharp,q_0$	♯	♯	♯	♯	♯	⋯

Algorithm and probability

An example of model of computation:



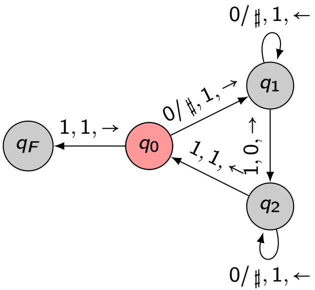
q_2

⋯	$\sharp$	$\sharp$	$\sharp$	1	1	1	0	1	$\sharp$	⋯

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	1, q_1	1	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$, q_0	1	1	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1, q_2	1	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0, q_2	1	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp$, q_2	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	1, q_1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0, q_0	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	1, q_2	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1, q_1	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp$, q_1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯
⋯	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$, q_0	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	⋯

Algorithm and probability

An example of model of computation:



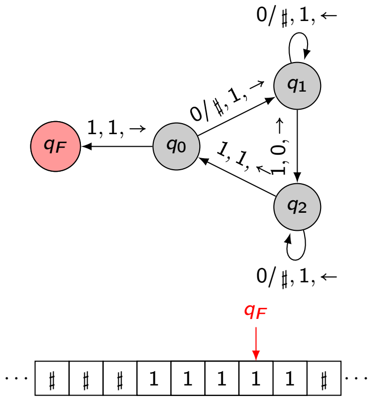
q_0

$\cdots$	$\sharp$	$\sharp$	$\sharp$	1	1	**1** (q_0)	1	1	$\sharp$	$\cdots$

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	1	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_2$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$0, q_2$	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp, q_2$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

Algorithm and probability

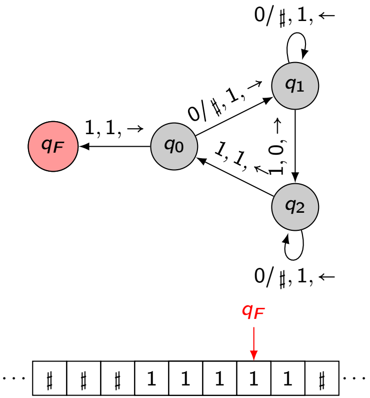
An example of model of computation:



	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	1	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_2$	1	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$0, q_2$	1	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	0	$\sharp, q_2$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$1, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$0, q_0$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	0	$1, q_2$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$1, q_1$	1	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	1	$\sharp, q_1$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp, q_0$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

Algorithm and probability

An example of model of computation:



	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	♯	♯	♯	♯	1	1, q_1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯, q_0	1	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_2	1	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0, q_2	1	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	0	♯, q_2	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	1, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0, q_0	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	0	1, q_2	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1, q_1	1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	1	♯, q_1	♯	♯	♯	♯	⋯
⋯	♯	♯	♯	♯	♯	♯, q_0	♯	♯	♯	♯	♯	⋯

Some probabilstic problematics:

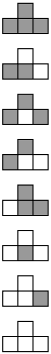
- What are the random initial configurations?
- What are the generic asymptotic behavior?
- What is the robustness to random perturbations?

Cellular Automata

Definition

- $\mathcal{A} = \{\square, \blacksquare\}$ finite *alphabet*
- $\mathcal{A}^{\mathbb{Z}^d}$ set of *configurations*
- $\overline{F} : \mathcal{A}^{\mathbb{U}} \longrightarrow \mathcal{A}$ *local rules*

$F(x)_{\mathbf{i}} = \overline{F}(x_{\mathbf{i}+\mathbb{U}})$ for all $x \in \mathcal{A}^{\mathbb{Z}^d}$



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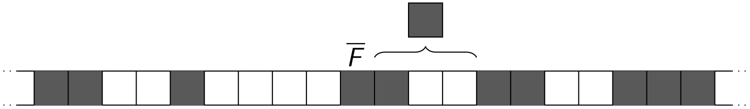

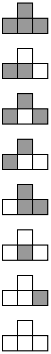

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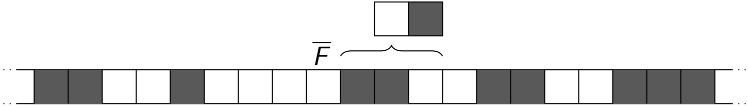

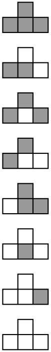

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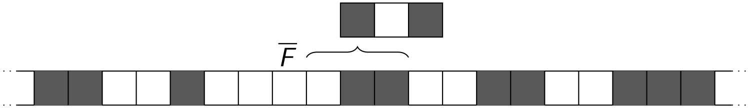

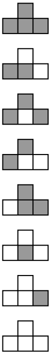

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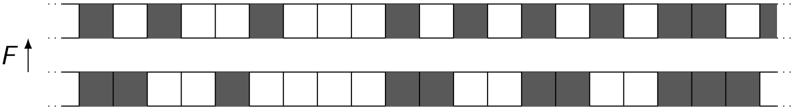

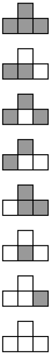

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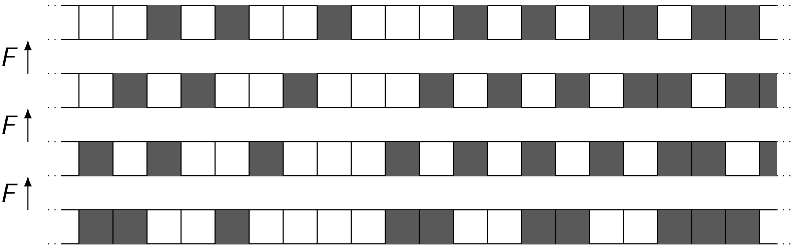
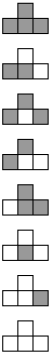
$$F(x)_{\mathbf{i}} = \overline{F}(x_{\mathbf{i}+\mathbb{U}}) \text{ for all } x \in \mathcal{A}^{\mathbb{Z}^{\boldsymbol{d}}}$$





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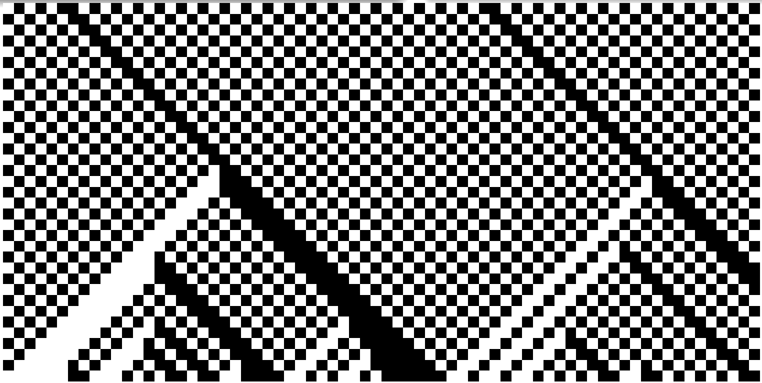
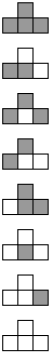
$F(x)_{\mathbf{i}} = \overline{F}(x_{\mathbf{i}+\mathbb{U}})$ for all $x \in \mathcal{A}^{\mathbb{Z}^{\mathbf{d}}}$

Theorem *(Hedlund-1969)*

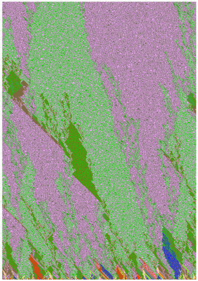
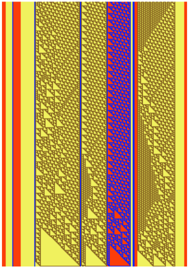
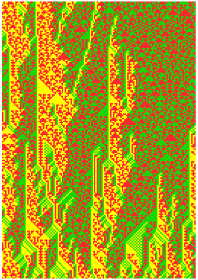
$(\mathcal{A}^{\mathbb{Z}}, F)$ is a CA iff $F : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ is continuous and $F \circ \sigma = \sigma \circ F$.

$$\begin{array}{rccc} \sigma : & \mathcal{A}^{\mathbb{Z}} & \longrightarrow & \mathcal{A}^{\mathbb{Z}} \\ & (x_i)_{i \in \mathbb{Z}} & \longmapsto & (x_{i+1})_{i \in \mathbb{Z}}. \end{array}$$

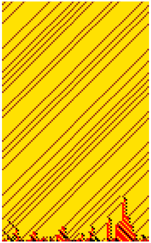
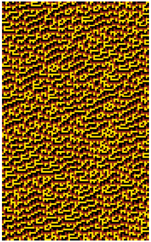
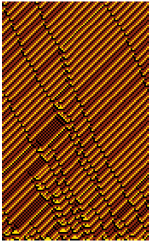




Some space-time diagrams

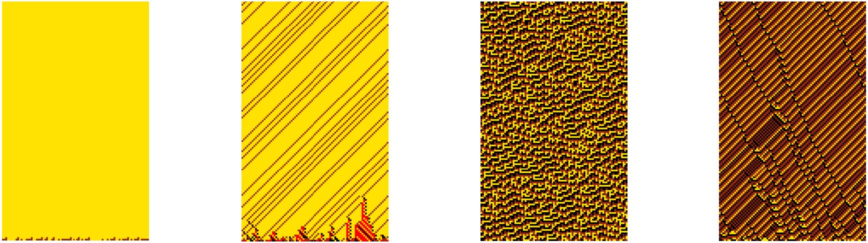




Classification of *Wolfram (1982)*:

Some space-time diagrams

Classification of *Wolfram (1982)*:

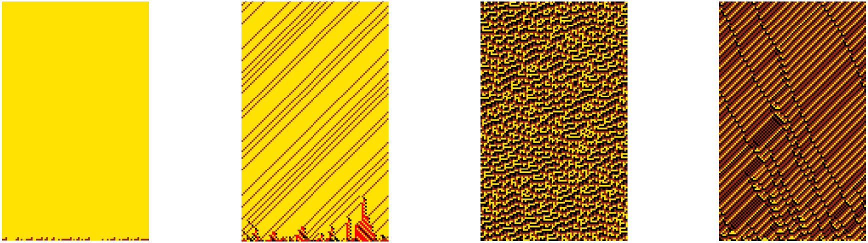


Highlighting and studying the propagation of information:

- Empirical approach

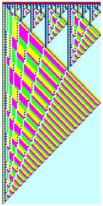
Some space-time diagrams

Classification of *Wolfram (1982)*:



Highlighting and studying the propagation of information:

- Empirical approach
- Algorithmically approach



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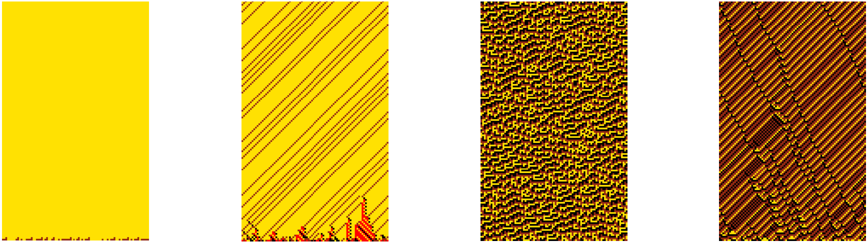

Highlighting and studying the propagation of information:

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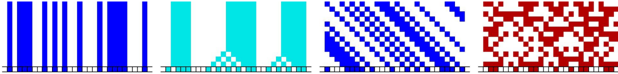
Classification of *Kurka*:



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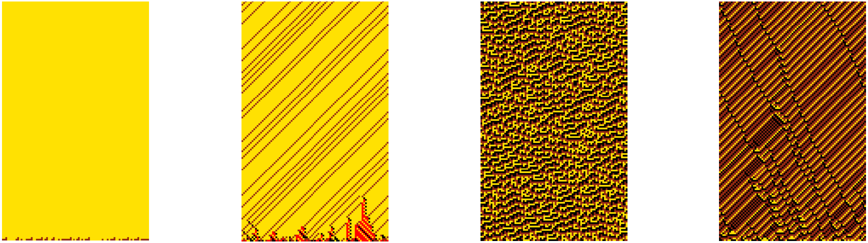

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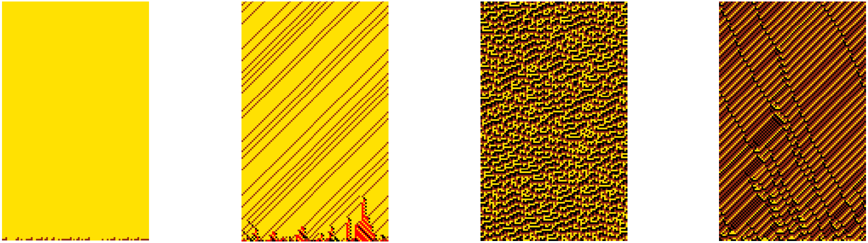

Highlighting and studying the propagation of information:

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Limit sets	**μ-limit sets**
$\Lambda(F) = \bigcap_{n\in\mathbb{N}} F^n(\mathcal{A}^{\mathbb{Z}})$	$u \notin \mathcal{L}(\Lambda_\mu(F))$ $\Leftrightarrow \lim_{n\to\infty} \mu(F^{-n}([u])) = 0$
$\Lambda\Big($  $\Big) =$	$\Lambda_\mu\Big($  $\Big) =$
$\left\{x \in \mathcal{A}^{\mathbb{Z}} : \square\blacksquare^n\square \not\sqsubset x\right\}$	$\begin{cases}\{^\infty\square^\infty\} & \text{if } \mu([\square]) > 0\}\\ \{^\infty\blacksquare^\infty\} & \text{if not}\end{cases}$

Different studies of the limit/μ-limit sets: *Hurley, Kari, Maass, Kurka, Theyssier...*

Some classes of measures: Dynamical properties

- Let $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ be the *set of σ-invariant probability measures*.
  Usually $\mathcal{M}_\sigma$ is endowed with the weak* topology:

$$\mu_n \underset{n\to\infty}{\longrightarrow} \nu \quad \text{iff} \quad \forall u \in \mathcal{A}^\mathbb{U} \text{ one has } \mu_n([u]) \underset{n\to\infty}{\longrightarrow} \nu([u]).$$

  $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is convex, compact and metrizable. $\forall \mu,\nu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ put:

$$d(\mu,\nu) = \sum_{n\in\mathbb{N}} \frac{1}{|\mathcal{A}|^n} \max_{u\in\mathcal{A}^n} |\mu([u]) - \nu([u])|.$$

- μ is *σ-ergodic* iff all σ-invariant subset $B \in \mathfrak{B}$, one has $\mu(B) = 0$ or 1.
  Consider $u, v \in \mathcal{A}^*$ and $x \in \mathcal{A}^\mathbb{Z}$.
  The *density* of u in v is $d_v(u) = \frac{\mathrm{Card}\{i\in[0,|u|-1]: v_{i,i+|u|-1}=u\}}{|v|-|u|+1}$.
  The *density* of u in x is $d_x(u) = \limsup_{n\to\infty} d_{x_{[-n,n]}}(u)$.

  We recall that for a σ-ergodic measure μ one has:

$$\mu([u]) = d_x(u) \text{ for } \mu\text{-almost all } x \in \mathcal{A}^\mathbb{Z}$$

Some classes of measures: Standard examples

- *Dirac measure*: for $x \in \mathcal{A}^{\mathbb{Z}}$ and $u \in \mathcal{A}^*$, $\delta_x([u]) = \begin{cases} 0 & \text{if } x \notin [u] \\ 1 & \text{if not} \end{cases}$
- *σ-invariant measure supported by a periodic point*:

$$\text{For } w \in \mathcal{A}^* \qquad \widehat{\delta_w} = \frac{1}{|w|} \sum_{i \in [1;|w|]} \delta_{\sigma^i({}^\infty w^\infty)}$$

Fact: $\left\{\widehat{\delta_w} : w \in \mathcal{A}^*\right\}$ *is dense in* $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$

- *Bernoulli measure* associated to $(p_a)_{a\in\mathcal{A}} \in [0;1]^{\mathcal{A}}$ such that $\sum_{a\in\mathcal{A}} p_a = 1$:

$$\lambda_{(p_a)_{a\in\mathcal{A}}}([u]) = p_{u_1} \dots p_{u_n} \text{ for } u = u_1 \dots u_n \in \mathcal{A}^*.$$

- *Markov measure*

Iteration of measures by a cellular automaton

$$\begin{array}{rccc} F: & \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) & \longrightarrow & \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \\ & \mu & \longmapsto & F\mu \end{array} \qquad \textit{such that } \forall B \in \mathfrak{B} \quad F\mu(B) = \mu(F^{-1}(B)).$$

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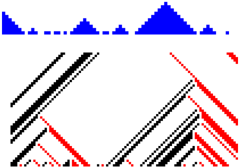

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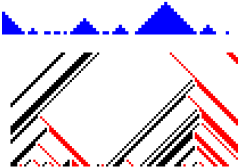

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$$F^n\mu \underset{n\to\infty}{\longrightarrow} \mu[\blacksquare]\widehat{\delta_\square} + \mu[\square]\widehat{\delta_{\textcolor{red}{\blacksquare}}} + \mu[\textcolor{red}{\blacksquare}]\widehat{\delta_\blacksquare} \quad \textit{(Hellouin)}$$

Iteration of measures by a cellular automaton

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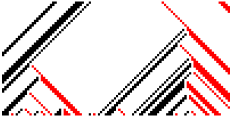

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$\frac{1}{n+1}\sum_{k=0}^{n} F^k\mu \underset{n\to\infty}{\longrightarrow} \lambda$ *(Maass-Martinez,Pivato-Yassawi...)*

Characterization of measure obtained asymptotically

Measure obtained asymptotically

- $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is *computable* if there exists $f : \mathcal{A}^* \times \mathbb{N} \to \mathbb{Q}$ computable such that
$$|\mu([u]) - f(u,n)| < \frac{1}{n}.$$
- $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is *limit-computable* if there exists $f : \mathcal{A}^* \times \mathbb{N} \to \mathbb{Q}$ computable such that
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Theorem of realization *(Hellouin-S.-14)*

Let $\nu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ be limit-computable, there exists $F : \mathcal{B}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z}$ with $\mathcal{A} \subset \mathcal{B}$
$$\forall\, \mu \in \mathcal{M}_{\mathrm{erg}}(\mathcal{B}^\mathbb{Z}) \qquad \mathrm{supp}(\mu) = \mathcal{B}^\mathbb{Z} \quad \Longrightarrow \quad F^n\mu \underset{n\to\infty}{\longrightarrow} \nu$$

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Keys of the construction:

ν limit-computable $\iff$ There exists a recursive sequence of words $(w_k)_{k\in\mathbb{N}}$ such that $\widehat{\delta}_{w_k} \xrightarrow[k\to\infty]{} \nu$.

$$\text{where } \widehat{\delta}_{w_k} = \frac{1}{|w_k|} \sum_{i=0}^{|w_k|-1} \delta_{\sigma^i({}^\infty w_k^\infty)}$$

Aim

We want to construct a cellular automaton which, starting from a μ-random configuration, generates successively $({}^\infty w_i^\infty)_{i\in\mathbb{N}}$.

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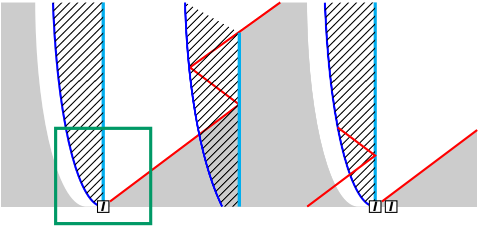
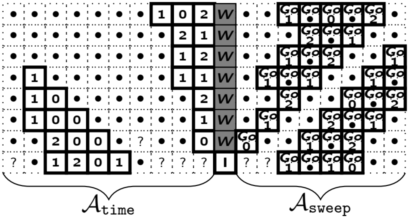
Keys of the construction:

1. *Formatting by absolute time counters*





- No transition rule produces the states $\boxed{\mathsf{I}}$
- $\boxed{\mathsf{I}}$ produces a time counter on the left and a sweeping counter on the right.

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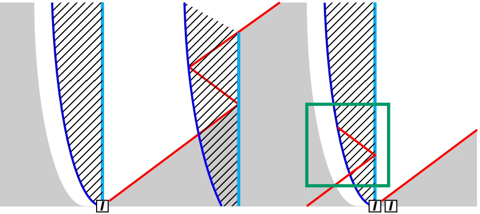

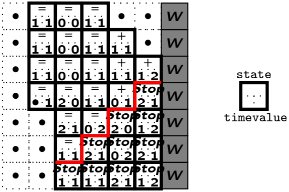

- During a collision, if the sweeping counter is older then it is deleted.

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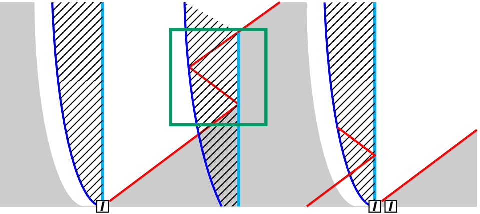

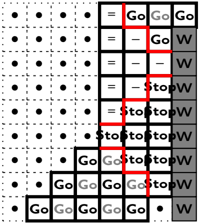

- During a collision, if the time counter is strictly older then it is deleted.

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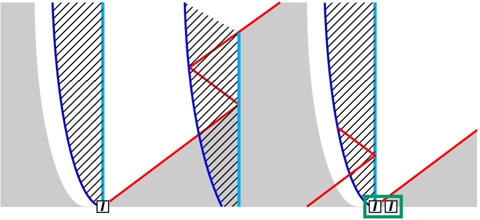

- States $\boxed{\text{I}}$ that are too close are deleted.

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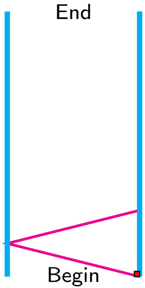
Keys of the construction:

1. *Formatting by absolute time counters*
2. *Computation and copy on segments*



- Compute the size of the segment k in $\log(k)$-space and allocate a space of size $\sqrt{k}$ for the computation;

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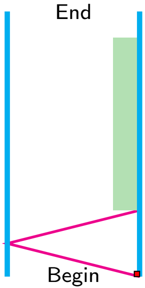

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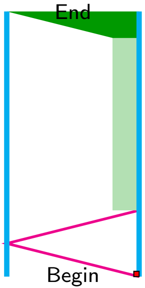

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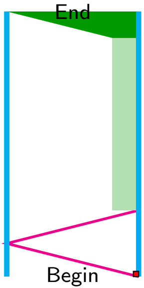

- Compute the size of the segment k in $\log(k)$-space and allocate a space of size $\sqrt{k}$ for the computation;
- Compute w_k;
- Copy periodically the word w_k on the segment.

$$d^t(u) \in \left[\min(d_{\mathrm{begin}}(u), d_{\mathrm{end}}(u)) - \frac{2\sqrt{k}}{k}; \max(d_{\mathrm{begin}}(u), d_{\mathrm{end}}(u)) + \frac{2\sqrt{k}}{k}\right]$$

$$\left|d_{\mathrm{end}}(u) - \widehat{\delta_{w_k}}([u])\right| \le \frac{\sqrt{k}}{k}$$

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Keys of the construction:

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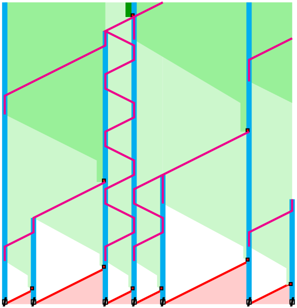
3. *Merging of two segments*



- Walls must disappear progressively in order to enlarge the computation zones and ensure that:

$$\mu \in \mathcal{M}_{\text{erg}}^{\text{full}}(\mathcal{B}^\mathbb{Z}) \quad \Longrightarrow \quad F^n\mu([\blacksquare]) \underset{n\to\infty}{\longrightarrow} 0$$

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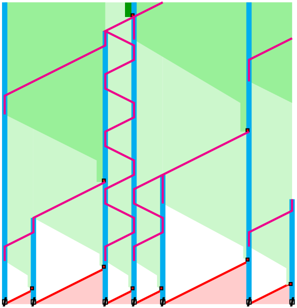
$$\forall \mu \in \mathcal{M}_{\mathrm{erg}}(\mathcal{B}^{\mathbb{Z}}) \qquad \mathrm{supp}(\mu) = \mathcal{B}^{\mathbb{Z}} \quad \Longrightarrow \quad F^n\mu \underset{n\to\infty}{\longrightarrow} \nu$$

Keys of the construction:



For $u \in \mathcal{A}^*$ and $\epsilon > 0$, one has:

- $\exists K \in \mathbb{N}$ such that $\forall k \geq K$, $|\widehat{\delta_{w_{\varphi(k)}}}([u]) - \nu([u])| < \epsilon$;
- $W_K(x) \subset \mathbb{Z}$: set of cells in segments larger than K.
  $\exists N \in \mathbb{N}$, $\forall n \geq N$, one has $d(W_K(x)) > 1 - \epsilon$ $\forall_{F^n\mu}\, x$;
- Let $r \in \mathrm{Conv}((\widehat{\delta_{w_{\varphi(i)}}}([u]))_{i\geq k})$.
  For enough large n, one has:

$$\begin{aligned} |F^n\mu([u]) - \nu([u])| &\leq |F^n\mu([u]) - r| + |r - \nu([u])| \\ &\leq \epsilon + \frac{4\sqrt{k}}{k} + \epsilon \end{aligned}$$

Thus $F^n\mu([u]) \underset{n\to\infty}{\longrightarrow} \nu([u])$

Which set of measures can be obtained asymptotically?

Some elements of computable analysis

- A *computable metric space* is a triple $(X, d, \mathcal{S})$, where
  - (X, d) is a compact metric space;
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- Notion of computability on X:
  - $x \in X$ is *computable* iff $\exists\ \alpha : \mathbb{N} \longmapsto \mathbb{N}$ computable such that $d(x, s_{\alpha(n)}) \leq 2^{-n}$.
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$$A_{\text{NoHalt}} = \{n : \text{the } n^{\text{th}} \text{ Turing machine does not halt on the empty entry}\}$$

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$$A_{\mathrm{Tot}} = \{n : \text{the } n^{\text{th}} \text{ Turing machine halts on every entry}\}$$

Computability obstructions

Let $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and consider

- $\mathcal{V}(F,\mu)$: set of cluster points of the sequence $(F^n\mu)_{n\in\mathbb{N}}$,
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$$\overline{B\left(\widehat{\delta_w}, r\right)} \cap \mathcal{V}(F,\mu) \neq \varnothing \iff \forall N,\ \exists n,\ d(F^n\mu, \widehat{\delta_w}) < r + 2^{-N}$$
$$\iff \inf_N \sup_n \alpha(N,n) = 1$$

where

$$\alpha(N,n) = \begin{cases} 1 & \text{if the approximation of } d(F^n\mu, \widehat{\delta_w}) \text{ at } 2^{-N} \text{ is less than } r + 2^{-N+1} \\ 0 & \text{otherwise} \end{cases}$$

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Proposition

If $\mathcal{V} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is compact, connected and Π_2-computable then there exists a computable sequence $(w_n)_{n\in\mathbb{N}}$ such that

$$\mathcal{V} = \bigcap_{n\in\mathbb{N}} \mathrm{Adh}\left(\bigcup_{i\geq n}\left[\widehat{\delta_{w_i}}, \widehat{\delta_{w_{i+1}}}\right]\right)$$

where $\left[\widehat{\delta_u}, \widehat{\delta_v}\right] = \left\{(1-t)\widehat{\delta_u} + t\widehat{\delta_v} : t \in [0,1]\right\}$ for all $u, v \in \mathcal{A}^*$.

Theorem of realization

Theorem of realization *(Hellouin-S.-14)*

Let $\nu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ be limit-computable, there exists $F : \mathcal{B}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z}$ with $\mathcal{A} \subset \mathcal{B}$

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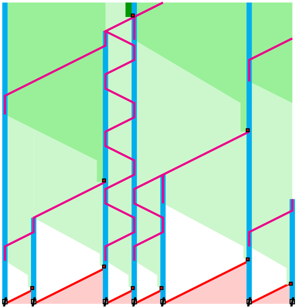
Keys of the construction:

1. *Formatting by absolute time counters*
2. *Computation and copy on segments*
3. *Merging of two segments*



Let $(w_i)_{i\in\mathbb{N}}$ be a recursive sequence of words, it is difficult to control the set of adherence value generated by the segments of different sizes.

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$$\forall\, \mu \in \mathcal{M}^{\mathrm{full}}_{\psi-\mathrm{mix}}(\mathcal{A}^{\mathbb{Z}}) \qquad \mathrm{supp}(\mu) = \mathcal{B}^{\mathbb{Z}} \quad \Longrightarrow \quad \mathcal{V}(F,\mu) = \bigcap_{n\in\mathbb{N}} \mathrm{Adh}\left(\bigcup_{i\geq n} \left[\widehat{\delta_{w_i}}, \widehat{\delta_{w_{i+1}}}\right]\right)$$

Keys of the construction:

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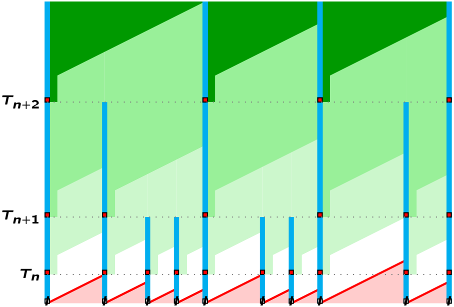

For $t \in [T_n, T_{n+1}]$ one has

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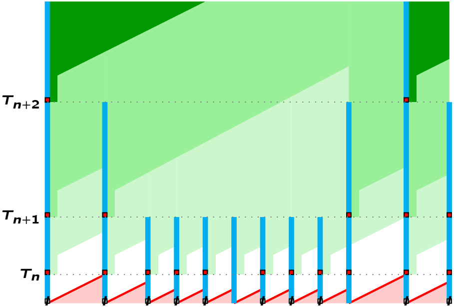

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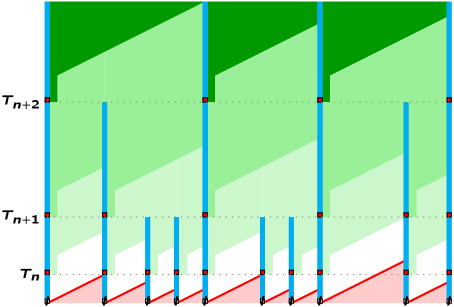

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This happens if $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{A}^{\mathbb{Z}})$

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Corollary *(Hellouin-S.-14)*

- $\mathcal{V}(F,\mu)$: set of cluster points of the sequence $(F^n\mu)_{n\in\mathbb{N}}$,
- $\mathcal{V}'(F,\mu)$: set of cluster points of the sequence $\left(\frac{1}{n}\sum_{k=0}^{n-1} F^k\mu\right)_{n\in\mathbb{N}}$.

Let $\mathcal{V}, \mathcal{V}' \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be compact, connected and Π_2-computable sets such that $\mathcal{V}' \subset \mathcal{V}$. There exists $(\mathcal{B}^{\mathbb{Z}}, F)$ such that for all $\mu \in \mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{B}^{\mathbb{Z}})$ one has:

$$\mathcal{V}(F,\mu) = \mathcal{V} \qquad \text{and} \qquad \mathcal{V}'(F,\mu) = \mathcal{V}'.$$

Some related questions

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Speed of convergence

Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of words on $\mathcal{A}$ computable in $O(\sqrt{n})$.
There exists $F : \mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ with $\mathcal{A} \subset \mathcal{B}$ such that

$$d(F^t\mu, \mathcal{V}((w_n)_{n\in\mathbb{N}})) = O\left(\frac{1}{\log(t)}\right) + \sup\left\{ \mathrm{d}_{\mathcal{M}}(\nu, \mathcal{V}((w_n)_{n\in\mathbb{N}})) \ : \ \nu \in \bigcup_{n \geq C(\log t)^2} [\widehat{\delta_{w_n}, \delta_{w_{n+1}}}] \right\}$$

where

$$\mathcal{V}((w_n)_{n\in\mathbb{N}}) = \bigcap_{N>0} \overline{\bigcup_{n\geq N} [\widehat{\delta_{w_n}, \delta_{w_{n+1}}}]}.$$

It is possible to improve the speed of convergence?

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Characterization of the set of limit points *(Hellouin & S.)*

Let $\mathcal{V}'\subset\mathcal{V}\subset\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ be connected Π_2-computable compact sets.
There exists $F:\mathcal{B}^{\mathbb{Z}}\to\mathcal{B}^{\mathbb{Z}}$ with $\mathcal{A}\subset\mathcal{B}$ such that

$$\forall\mu\in\mathcal{M}^{\text{full}}_{\psi-\text{mix}}(\mathcal{B}^{\mathbb{Z}}),\quad \mathcal{V}(F,\mu)=\mathcal{V}\text{ and }\mathcal{V}'(F,\mu)=\mathcal{V}'.$$

Rice Theorem *(Hellouin & S.)*

Any non-trivial property on asymptotic set of measures is undecidable.

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Theorem *(Hellouin & S.)*

Let $u \in \mathcal{A}^*$ be a word that does not appear in the support of $\mathcal{V} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$. There is a CA $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that

$$\forall\, \mu \in \mathcal{M}^{\mathrm{full}}_{\psi-\mathrm{mix}}(\mathcal{A}^{\mathbb{Z}}), \quad \mathcal{V}(F,\mu) = \mathcal{V}.$$

But if $F : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ is surjective, we have additional obstructions:

- $F\lambda_{\mathcal{A}^{\mathbb{Z}}} = \lambda_{\mathcal{A}^{\mathbb{Z}}}$ *(Hedlund-1969)*;
- $h_{F\mu}(\sigma) = h_\mu(\sigma)$ *(Kari-Taati-2014)* so $\mathcal{V}(F,\mu) \subset \{\nu : h_\nu(\sigma) \geq h_\mu(\sigma)\}$.

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- *What happens for multidimensional CA?* (Ask Hellouin and Delacourt)
- *What is the speed of convergence?* (Ask Hellouin)
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An operator $\varphi : \mathcal{M} \longrightarrow \mathcal{M}'$ is *limit computable* if there exists $f : \mathcal{M} \times \mathcal{A}^* \times \mathbb{N} \to \mathbb{Q}$ a computable function with oracle such that $f(\mu, u, n) \underset{n\to\infty}{\longrightarrow} \varphi(\mu)([u])$.

Computability obstruction: If $(F^n\mu)_n$ converges for all $\mu \in \mathcal{M}_\sigma \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ $\mu \mapsto \lim_{n\to\infty} F^n\mu$ is a limit-computable operator.

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Theorem *(Hellouin & S.)*

Let $\varphi : \mathcal{M}_{\psi-\exp}(\{0,1\}^\mathbb{Z}) \longrightarrow \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ be a limit computable operator, there exists a cellular automaton $(\mathcal{B}^\mathbb{Z}, F)$ and a factor $\pi : \mathcal{B} \longrightarrow \{0,1\}$ such that $\mathcal{V}(F,\mu) = \{\varphi(\pi\mu)\}$ for all $\mu \in \mathcal{M}_{\psi-\exp}(\mathcal{B}^\mathbb{Z})$ of full support.

Some related questions

- *What happens for multidimensional CA?* (Ask Hellouin and Delacourt)
- *What is the speed of convergence?* (Ask Hellouin)
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Perturbation of a cellular automaton F by a random noise R_ϵ

Limit measures are conjectured algorithmically simple. Some idea in this sense:

- Studies for the transformation of the interval *(Braverman-Grigo-Rojas-2013)*;
- It is true for large class of exemples (*Marcovici-S.-Taati-2016*) for exemple:

$$\text{If } F \text{ is surjective, } (R_\epsilon \circ F)^n \mu \underset{n\to\infty}{\longrightarrow} \lambda_{\mathcal{A}^{\mathbb{Z}}}.$$

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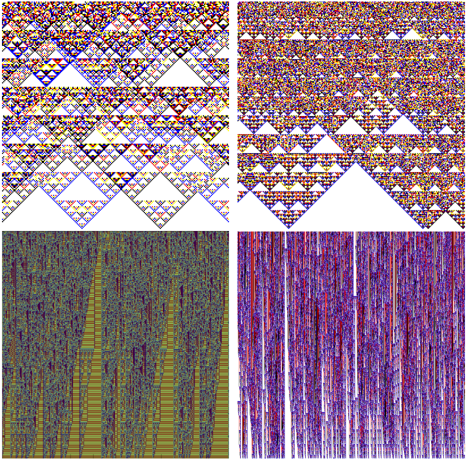
$$\frac{1}{n}\sum_{k=0}^{n-1} F^k \mu \underset{n\to\infty}{\longrightarrow} \lambda_{\mathcal{A}^{\mathbb{Z}}}$$



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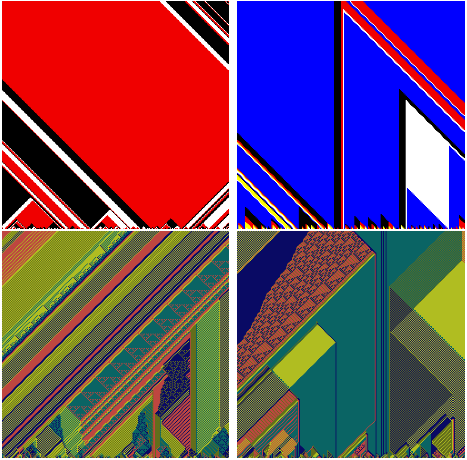

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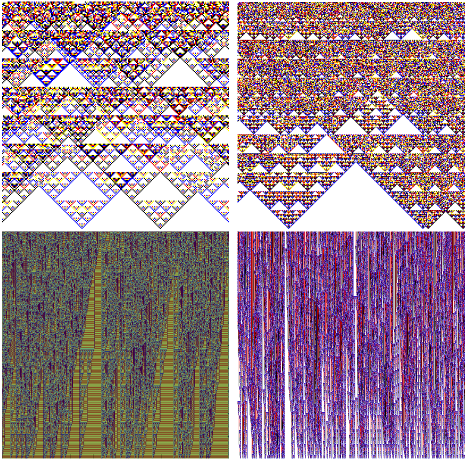

- Randomization results

▸ *Lind-84* shows that $((\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}, \mathrm{Id}+\sigma)$ randomizes Bernoulli measures

▸ Randomization for large classes of Algebraic CA and initial measures.

Two approaches:

- stochastic processes *Ferrari-Maass-Martínez-Ney-00*
- Harmonic analysis *Pivato-Yassawi-02*

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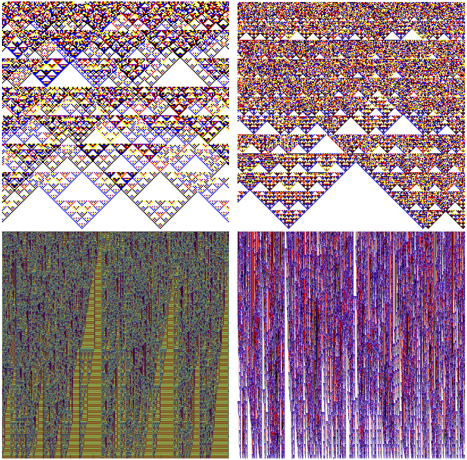

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- *Rigidity results*

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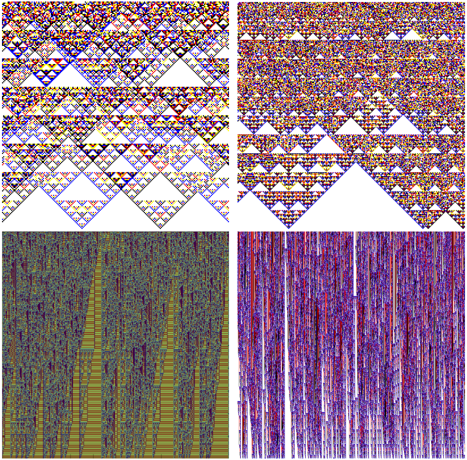

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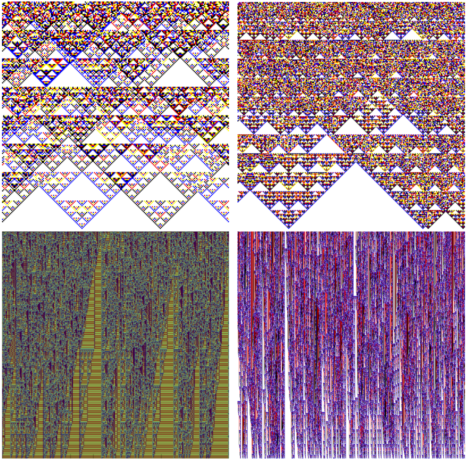

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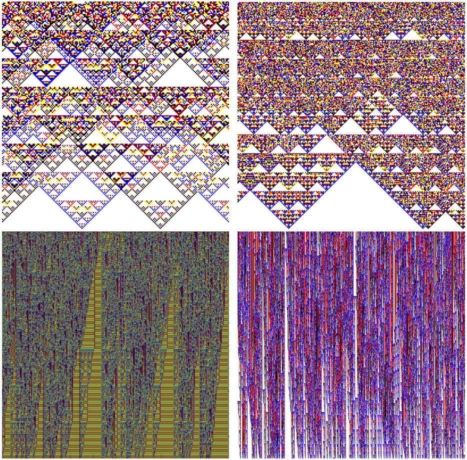

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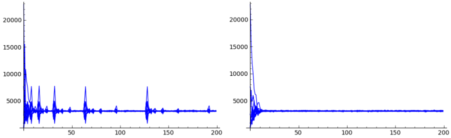

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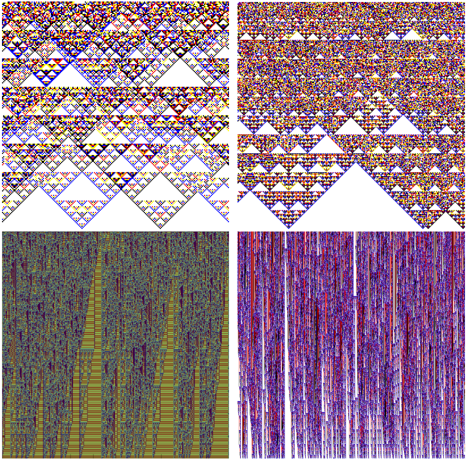

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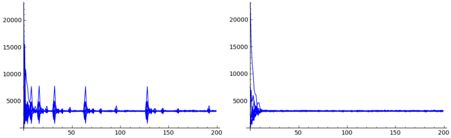

Challenging question

Prove randomization or rigidity results for expansive cellular automata.

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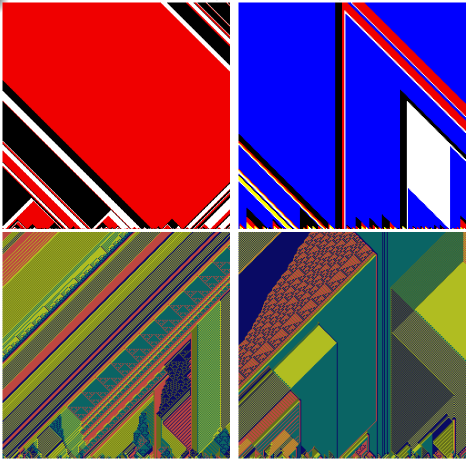

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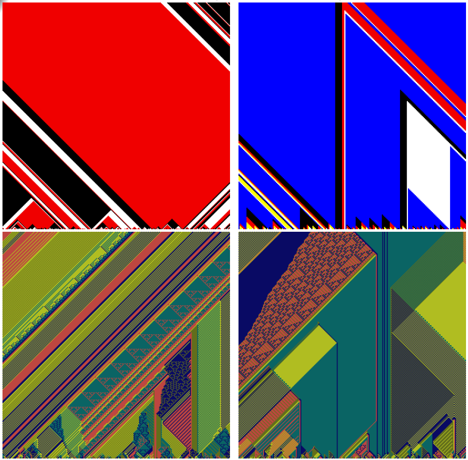
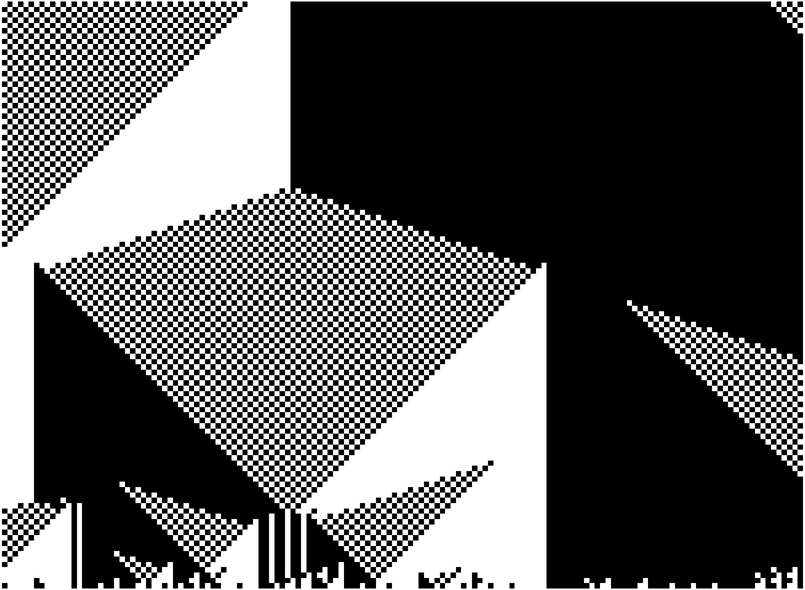


Homogeneous regions vs defects



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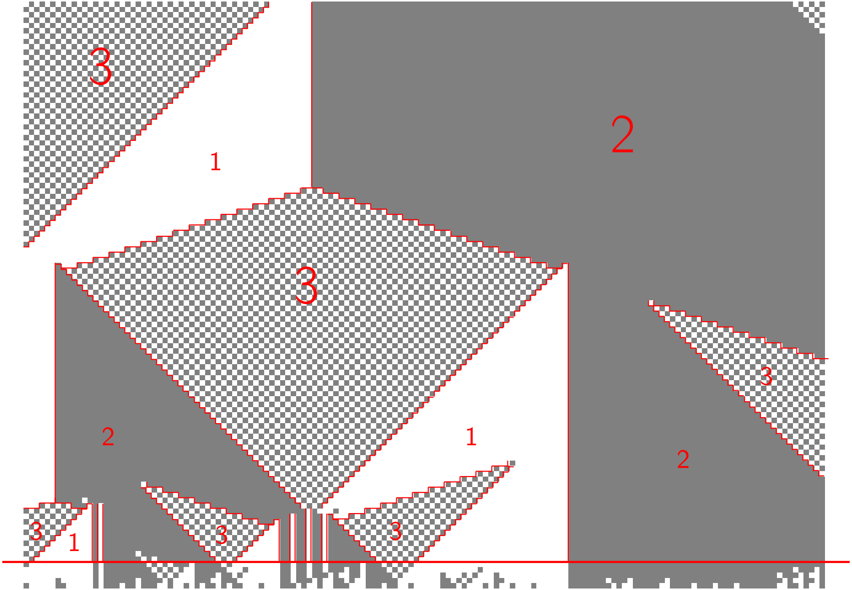

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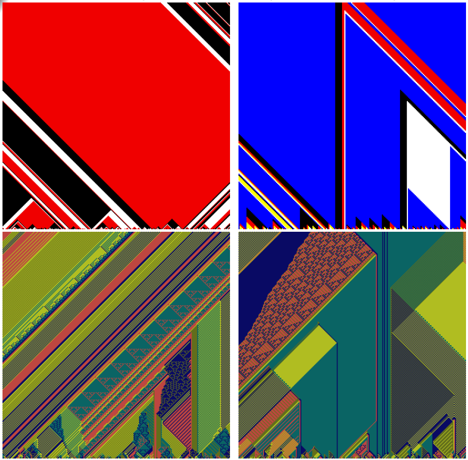

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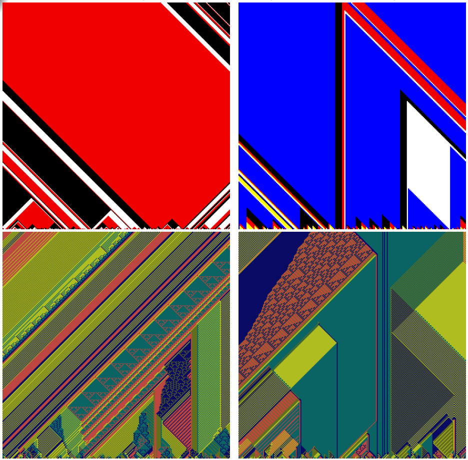

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- *Quantitative approach:*
  - "gliders CA": precise description of distribution of particles *[HS12]*

$$\mu\left(\frac{T_n^-(a)}{n} \leq x\right) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi} \arctan\left(\sqrt{\frac{-v_- x}{v_+ - v_- + v_+ x}}\right)$$

where $T_n^-(x) = \min\left\{k \in \mathbb{N} \mid F^{k+n}(x)_0 = -1\right\}$

Principal tool: Brownian motion

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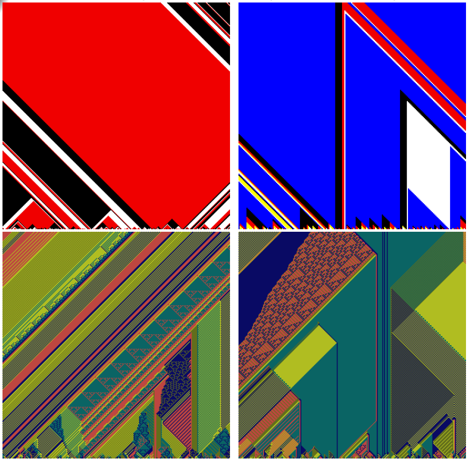

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