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Processus Ponctuel de β-Ginibre et Déploiement d'un Réseau Cellulaire

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Introduction

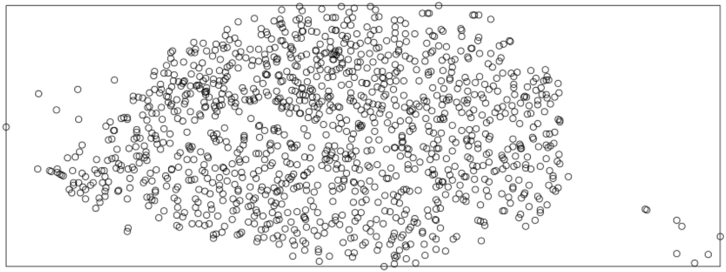

Figure: Positions of all base stations in Paris.

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Framework

- $\mathbb{Y}$ a locally compact metric space
- μ a diffuse and locally finite measure of reference on $\mathbb{Y}$
- $N_{\mathbb{Y}}$ the space of configurations on $\mathbb{Y}$
- $\widehat{N}_{\mathbb{Y}}$ the space of finite configurations on $\mathbb{Y}$

Determinantal point process

Definition

- **Correlation function** ρ of a point process Φ:

$$\boxed{\mathbb{E}[\sum_{\substack{\alpha \in \widehat{N}_{\mathbb{Y}} \\ \alpha \subset \Phi}} f(\alpha)] = \sum_{k=0}^{+\infty} \frac{1}{k!} \int_{\mathbb{Y}^k} f \cdot \rho(\{x_1, \ldots, x_k\}) \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_k)}$$

$\rho(\alpha) \approx$ probability of finding a point in at least each point of α

Definition

Determinantal point process DPP(C, μ):

$$\rho(\{x_1, \cdots, x_k\}) = \det(C(x_i, x_j), \ 1 \leq i, j \leq k)$$

β-Ginibre point process (1/2)

Definition

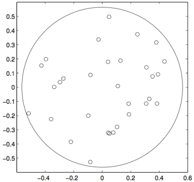
- Ginibre point process with intensity $\lambda = \dfrac{\gamma}{\pi}$:

$$C_\gamma(x,y) = \frac{\gamma}{\pi} e^{-\frac{\gamma}{2}(|x|^2+|y|^2)} e^{\gamma x \bar{y}}$$

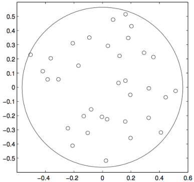
- β-Ginibre point process with intensity $\lambda = \dfrac{\gamma}{\pi}$:

$$C_{\gamma,\beta}(x,y) = \frac{\gamma}{\pi} e^{-\frac{\gamma}{2\beta}(|x|^2+|y|^2)} e^{\frac{\gamma}{\beta} x \bar{y}}$$

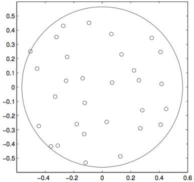
β-Ginibre point process (2/2)



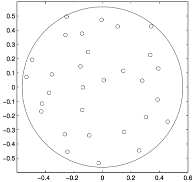
(a) PPP



(b) ¼-GPP



(c) ¾-GPP



(d) 1-GPP

J-function (1/2)

Definition

- Φ a stationary point process
- Empty space function:

$$F(r) = \mathbb{P}(\|u - \Phi\| \leq r)$$

- Nearest-neighbor distance distribution function:

$$G(r) = \mathbb{P}(\|u - \Phi \setminus \{u\}\| \leq r)$$

- J-function:

$$J(r) = \frac{1 - G(r)}{1 - F(r)}$$

J-function (2/2)

Interpretation

- $J < 1 \implies \Phi$ is attractive
- $J > 1 \implies \Phi$ is repulsive
- $J = 1 \implies \Phi$ is a Poisson point process

Proposition (Haenggi et al.)

- J-function of a β-Ginibre point process:

$$J(r) = \frac{1}{1 - \beta + \beta e^{-\frac{\gamma}{\beta} r^2}}$$

Fitting method

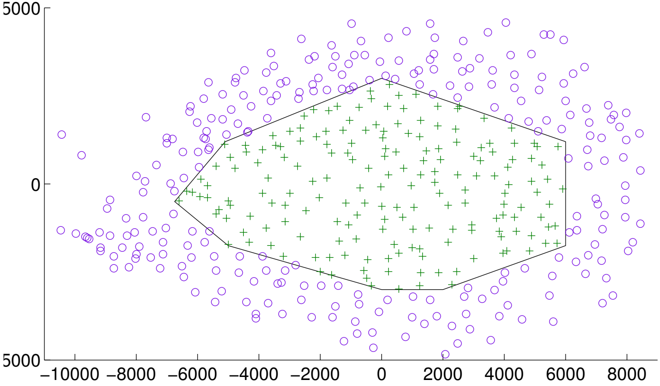


Figure: Example of data sample for one GSM operator. The *J*-function is fitted on the points within the rectangular window.

Example of J-function fitting

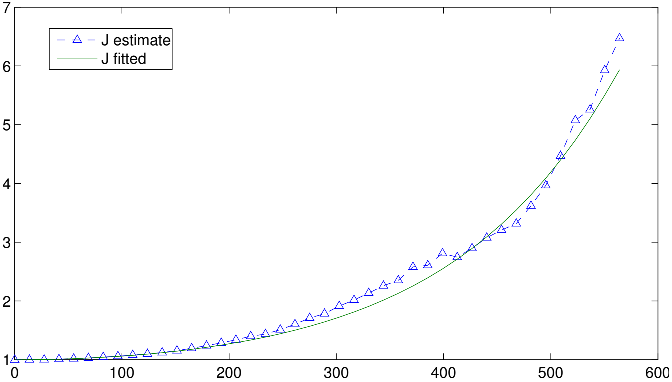


Figure: Example of J-function fitting for SFR on the 3G 900 MHz band. $\lambda = 1.92$ base stations per kilometre square $\beta = 0.97$. Residuals: 0.74.

Results per technology and operator

Table: Numerical values of β per technology and operator.

	Orange	SFR	Bouygues	Free
GSM 900	0.81	0.76	0.65	NA
GSM 1800	0.84	0.85	0.71	NA
UMTS 900	NA	0.97	0.53	0.89
UMTS 2100	1.04	0.65	0.82	0.89
LTE 800	1.02	0.93	0.67	NA
LTE 1800	NA	NA	0.75	NA
LTE 2600	0.93	0.67	0.63	0.89

Results per operator and for the superposition

Table: Numerical values of β and λ per operator and for the superposition of all the sites.

	Orange	SFR	Bouygues	Free	**Superposition**
β	0,94	0,70	0,81	0,89	**0,17**
λ	3,48	3,70	4,23	1,05	**10,28**
Number of sites	185	197	225	56	**547**

Papangelou intensity

Definition

- **Papangelou intensity c** of a point process Φ:

$$\boxed{\mathbb{E}[\sum_{x\in\Phi} f(x,\Phi\setminus\{x\})] = \int_{\mathbb{Y}} \mathbb{E}[c(x,\Phi)f(x,\Phi)]\mu(\mathrm{d}x)}$$

$c(x,\xi) \approx$ conditionnal probability of finding a point in x given ξ

Proposition (Georgii et al.)

Papangelou intensity of $\mathrm{DPP}(C,\mu)$:

$$c(x_0,\{x_1,\cdots,x_k\}) = \frac{\det(H(x_i,x_j),\ 0\leq i,j\leq k)}{\det(H(x_i,x_j),\ 1\leq i,j\leq k)}$$

where $H=(I-C)^{-1}C$.

Repulsive point process

Definition

- Point process repulsive if

$$\phi \subset \xi \implies c(x, \xi) \leq c(x, \phi)$$

- Point process weakly repulsive if

$$c(x, \xi) \leq c(x, \emptyset)$$

Proposition (Georgii et al.)

- DPP(C, μ) is repulsive.

Kantorovich-Rubinstein distance

- Total variation distance:

$$d_{\mathrm{TV}}(\nu_1,\nu_2) := \sup_{\substack{A\in\mathcal{F}_{\mathbb{Y}} \\ \nu_1(A),\nu_2(A)<\infty}} |\nu_1(A)-\nu_2(A)|$$

- $F : N_{\mathbb{Y}} \to \mathbb{R}$ is 1-Lipschitz ($F \in \mathrm{Lip}_1$) if

$$|F(\phi_1) - F(\phi_2)| \leq \mathrm{d}_{TV}(\phi_1,\phi_2) \text{ for all } \phi_1,\phi_2 \in N_{\mathbb{Y}}$$

- Kantorovich-Rubinstein distance:

$$d_{\mathrm{KR}}(\mathbb{P}_1,\mathbb{P}_2) = \sup_{F\in\mathrm{Lip}_1} \left| \int_{N_{\mathbb{Y}}} F(\phi)\,\mathbb{P}_1(\mathrm{d}\phi) - \int_{N_{\mathbb{Y}}} F(\phi)\,\mathbb{P}_2(\mathrm{d}\phi) \right|$$

- Convergence in K.-R. distance $\underset{strictly}{\Longrightarrow}$ Convergence in law

Upper bound theorem

Theorem (LD, AV)

- *Φ a finite point process on $\mathbb{Y}$*
- *ζ_M a PPP with finite control measure $M(dy) = m(y)\mu(dy)$.*

Then, we have:

$$d_{KR}(\mathbb{P}_{\Phi}, \mathbb{P}_{\zeta_M}) \leq \int_{\mathbb{Y}} \mathbb{E}[|m(y) - c(y, \Phi)|]\mu(dy).$$

Superposition of weakly repulsive point processes

- $\Phi_{n,1}, \dots, \Phi_{n,n}$: n independent, finite and **weakly repulsive** point processes on $\mathbb{Y}$
- $\Phi_n := \sum_{i=1}^{n} \Phi_{n,i}$
- $R_n := \int_{\mathbb{Y}} |\sum_{i=1}^{n} \rho_{n,i}(x) - m(x)|\mu(\mathrm{d}x)$
- ζ_M a PPP with control measure $M(\mathrm{d}x) = m(x)\mu(\mathrm{d}x)$

Superposition of weakly repulsive point processes

Proposition (LD, AV)

- $\Phi_n = \sum_{i=1}^{n} \Phi_{n,i}$
- ζ_M a PPP with control measure $M(\mathrm{d}x) = m(x)\mu(\mathrm{d}x)$

$$d_{\mathrm{KR}}(\mathbb{P}_{\Phi_n}, \mathbb{P}_{\zeta_M}) \leq R_n + \max_{1\leq i\leq n} \int_{\mathbb{Y}} \rho_{n,i}(x)\mu(\mathrm{d}x)$$

β-Ginibre point processes

Proposition (LD, AV)

- Φ_n the β_n-Ginibre process reduced to a compact set Λ
- ζ the PPP with intensity $1/\pi$ on Λ

$$d_{\mathrm{KR}}(\mathbb{P}_{\Phi_n}, \mathbb{P}_{\zeta}) \leq \frac{2}{\pi} \frac{\beta_n}{1-\beta_n} |\Lambda|$$

References

- J.S. Gomez, A. Vasseur, A. Vergne, P. Martins, L. Decreusefond, and W. Chen, A Case Study on Regularity in Cellular Network Deployment, IEEE Wireless Communications Letters, 2015.
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- N. Deng, W. Zhou, and M. Haenggi, The Ginibre Point Process as a Model for Wireless Networks with Repulsion, 2014.
- H.O. Georgii, and H.J. Yoo, Conditional intensity and gibbsianness of determinantal point processes, J. Statist. Phys. (118), 2004.

Thank you ...

... for your attention. Questions?