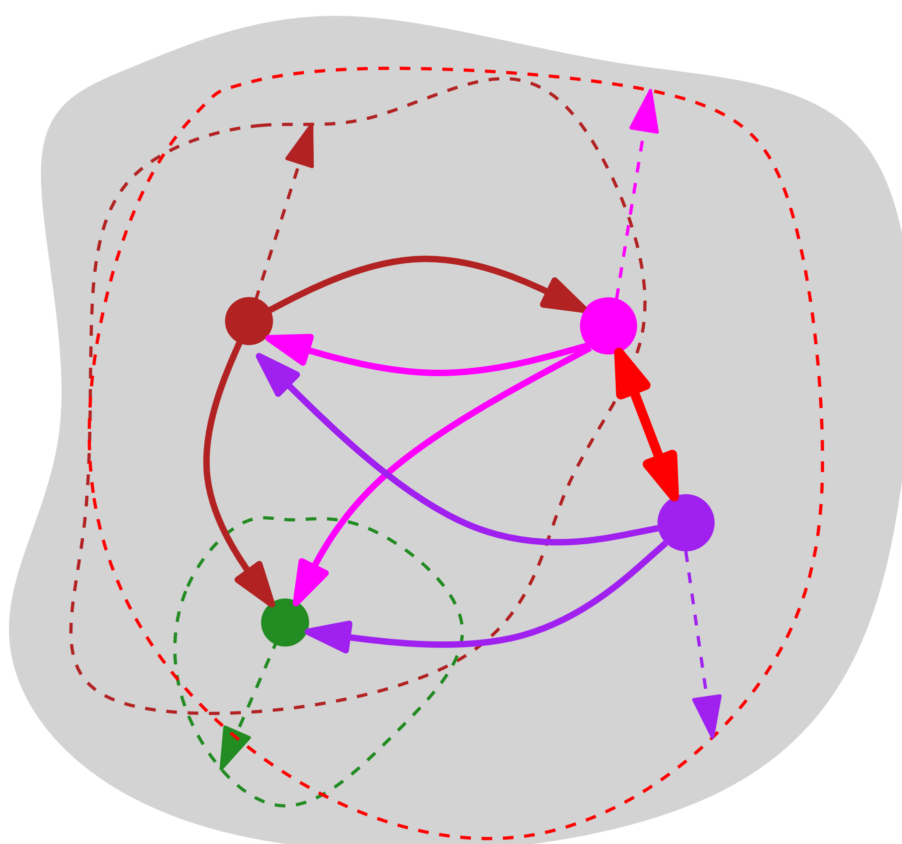
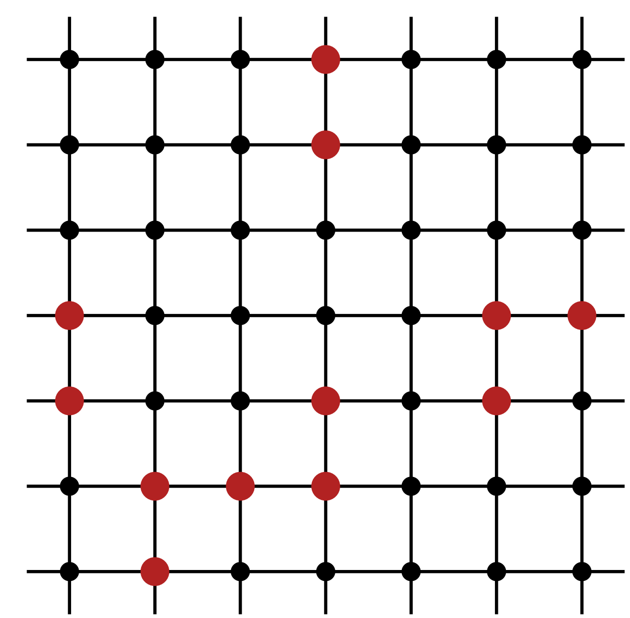
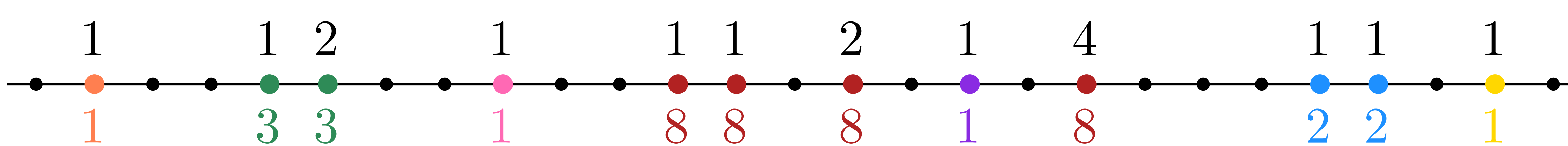
Percolation by cumulative merging and phase transition of the contact process

Laurent Ménard (Modal'X)

joint work with **Arvind Singh** (Orsay)

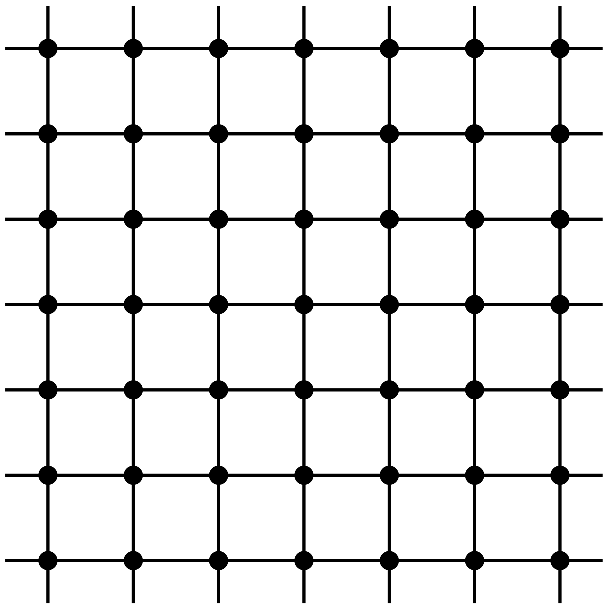






The contact process (Susceptible-Infected-Susceptible)

Epidemic model on graphs introduced by [Harris 74]



$G = (V, E)$ locally finite graph, $\lambda > 0$.

- Vertices are either **healthy** or **infected**.
- An infected site **recovers** at rate 1.
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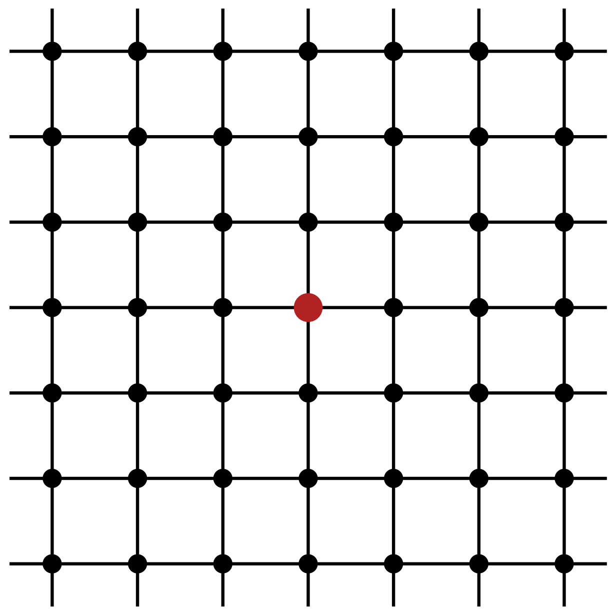

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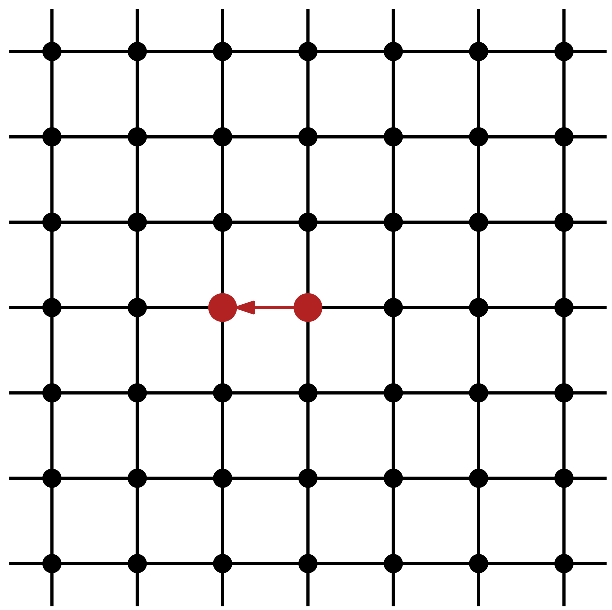

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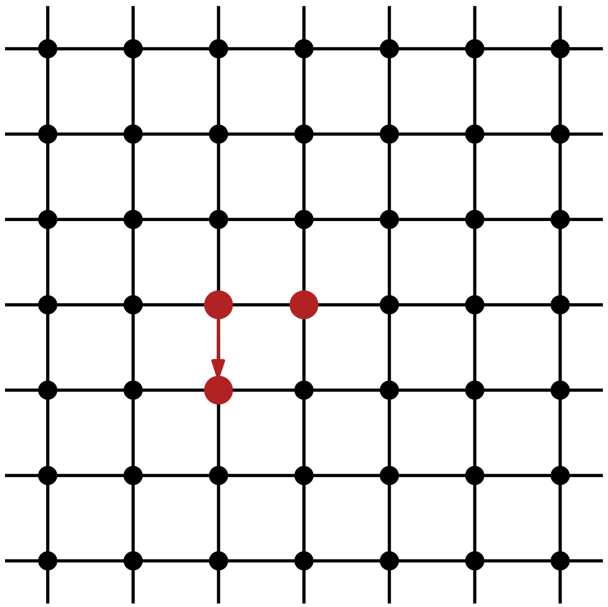

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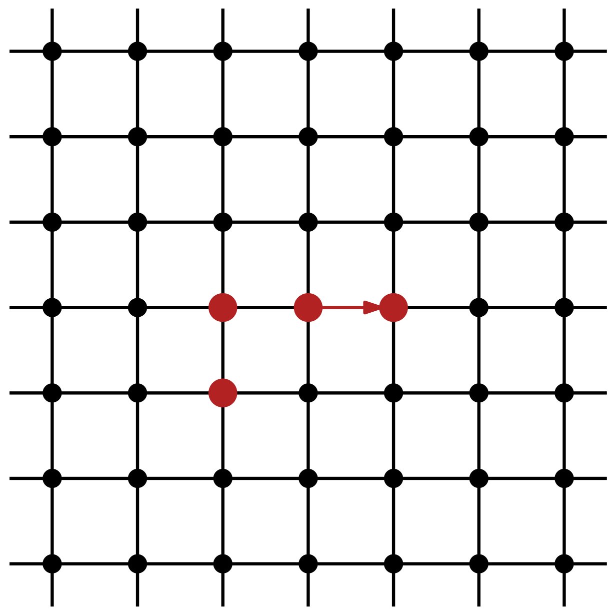

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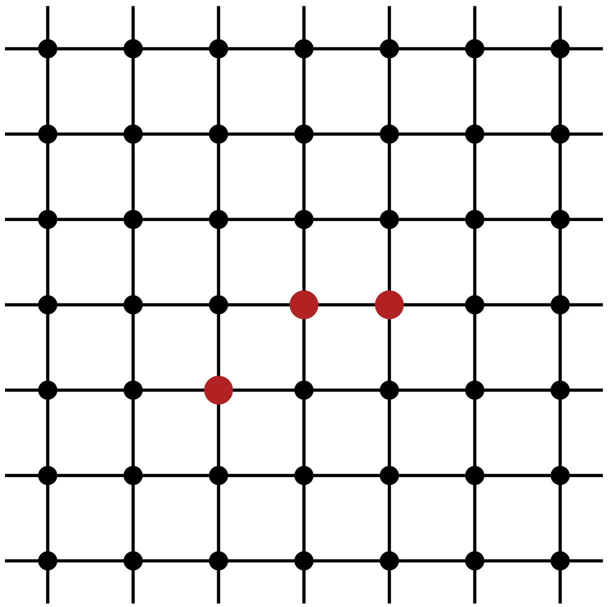

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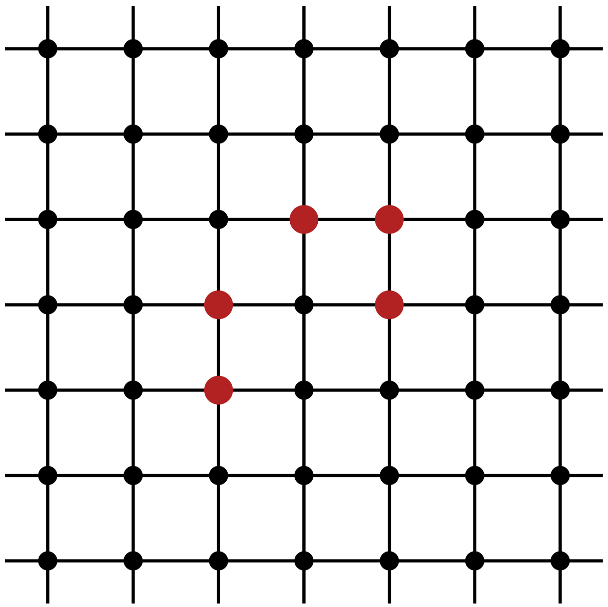

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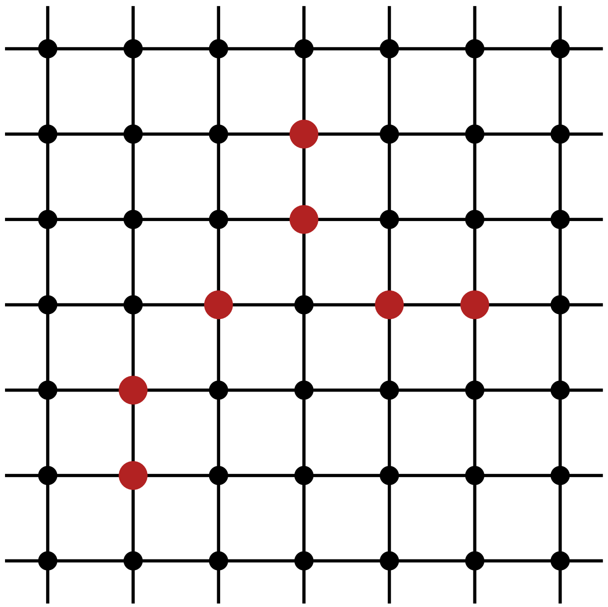

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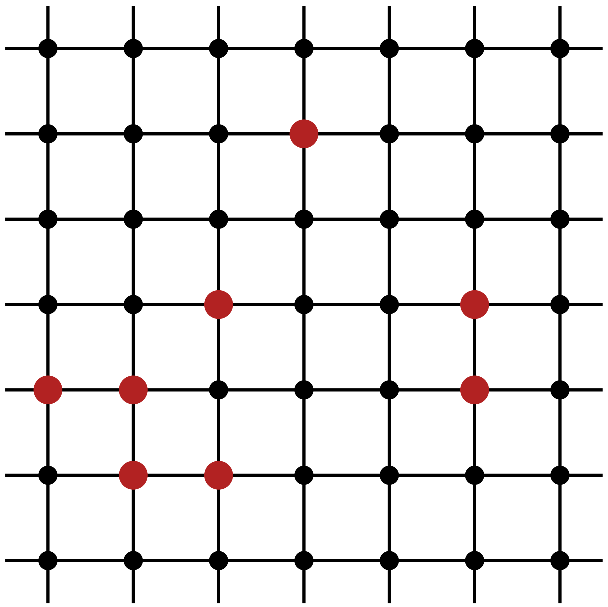

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On an infinite graph, **phase transition**: there is $\lambda_c \in [0, \infty[$ such that

- if $\lambda < \lambda_c$, the infection **dies out** *a.s.*;
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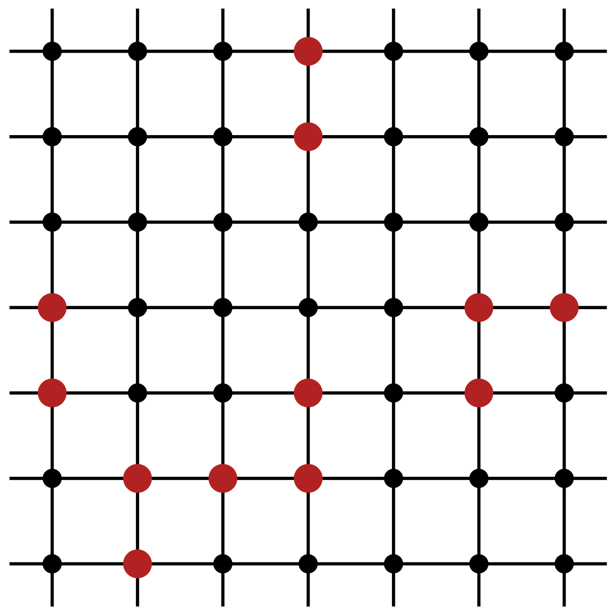

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Question: condition on G to ensure $\lambda_c > 0$?

The contact process on a graph with bounded degrees

If G has **bounded** degrees, then $\lambda_c > 0$.

Compare with **branching random walk**:

- **No interaction** between particles;
- particles die at rate 1;
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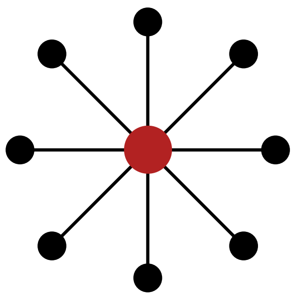
- No other method to prove that contact process dies out.
- No example of graph with unbounded degrees for which we know $\lambda_c > 0$.

Heuristics for the contact process

Contact process on a star graph of large degree d:



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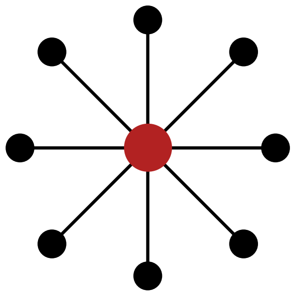

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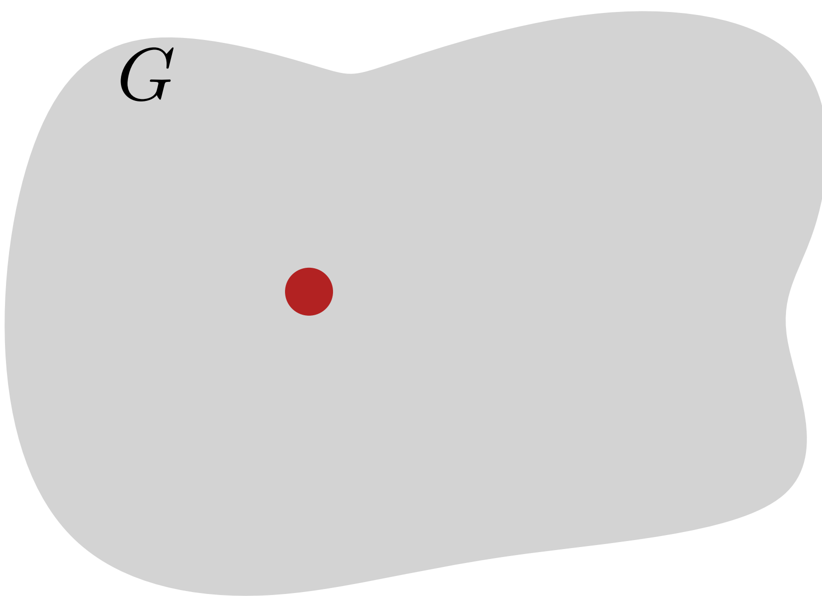
Now fix $\lambda > 0$ and consider contact process on a gaph G *s.t.*

- One vertex (●) has large degree d_0 with $\lambda > \lambda_c(d_0)$;
- all other vertices have small degrees d with $\lambda \ll \lambda_c(d)$.



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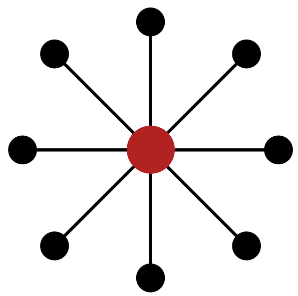

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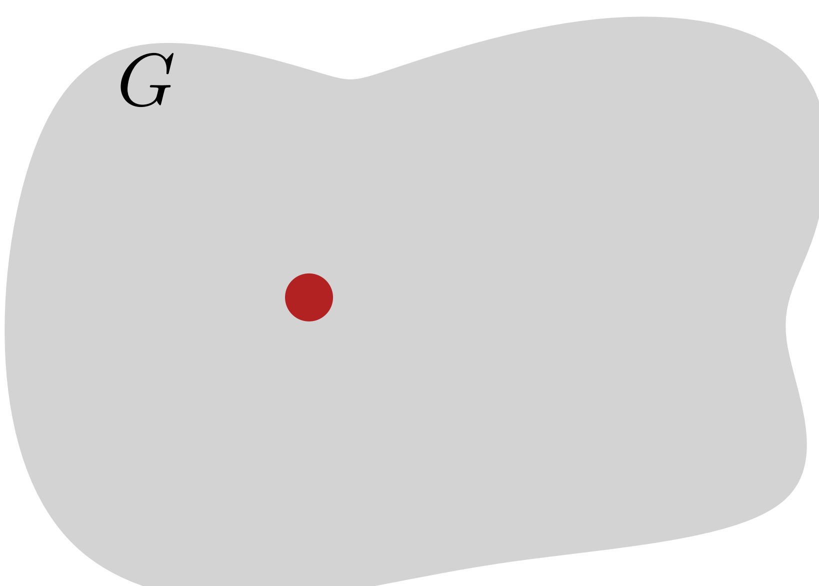

- Start with only ● infected.
- Force ● to stay infected a time $\exp(d_0)$.
- After that time, force the whole star around ● to recover.

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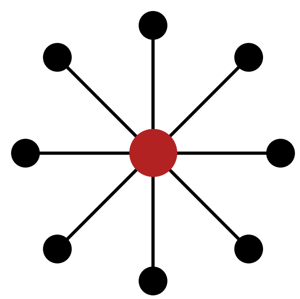

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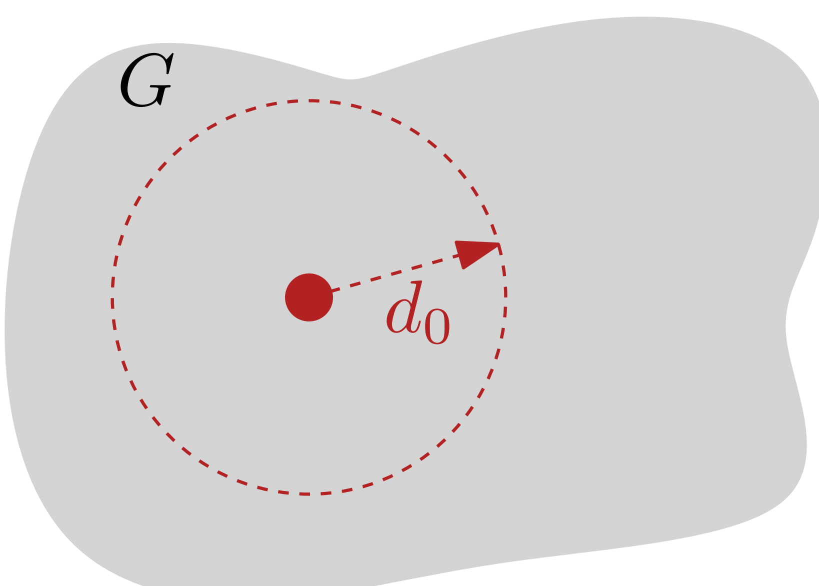

- Start with only ● infected.
- Force ● to stay infected a time $\exp(d_0)$.
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Maximal distance reached by infection is $\approx d_0$.

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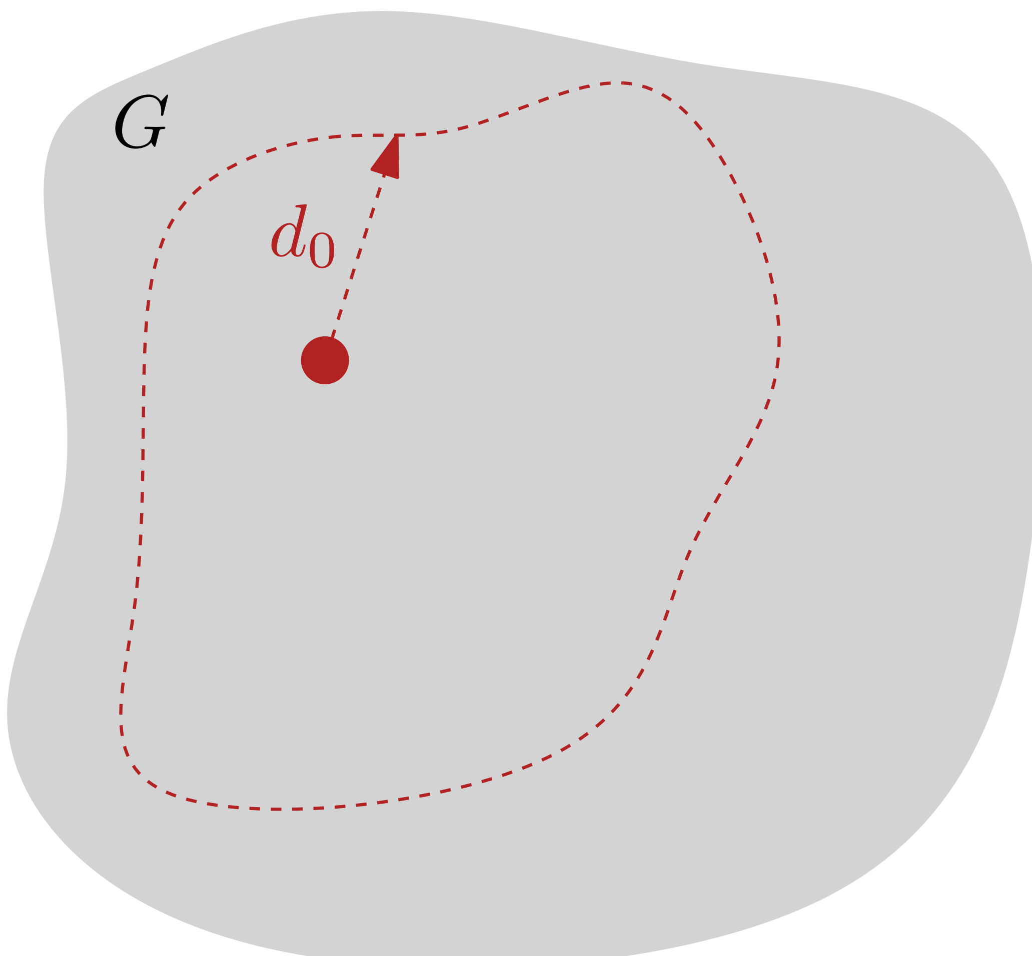
Same as before for ● and start of the infection.



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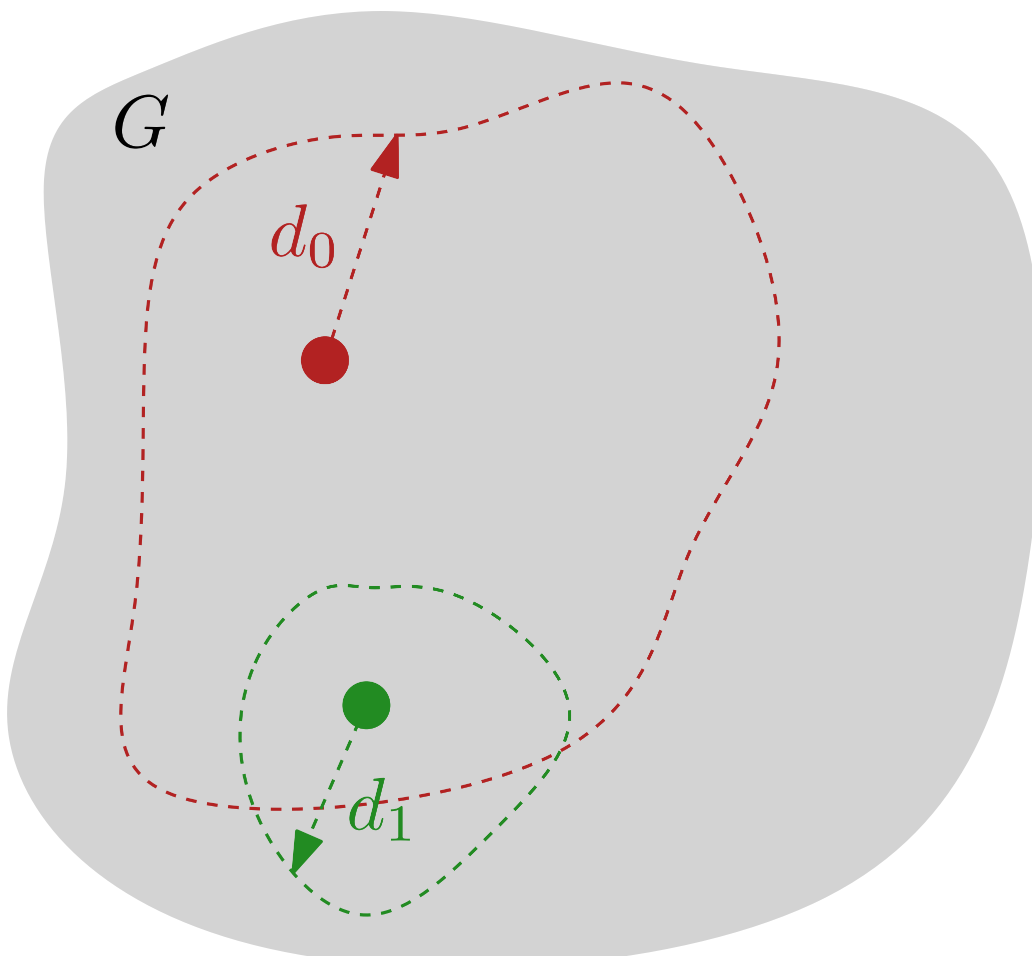

In addition suppose:

- At distance $< d_0$ from ● , there is another vertex (●) with large degree d_1 *s.t.* $\lambda > \lambda_c(d_1)$.
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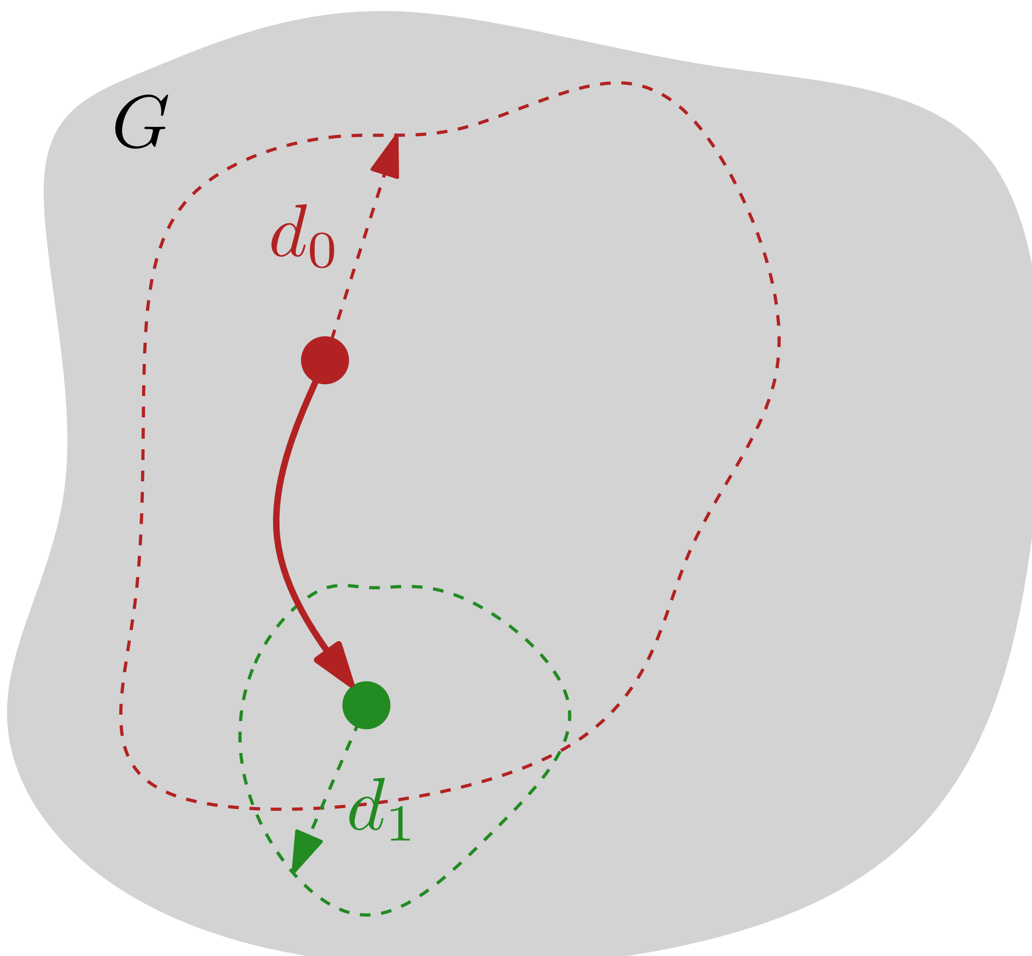

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● cannot send infections to ● and the survival time of the process is $\approx \exp(d_0) + \exp(d_1) \approx \exp(d_0)$.

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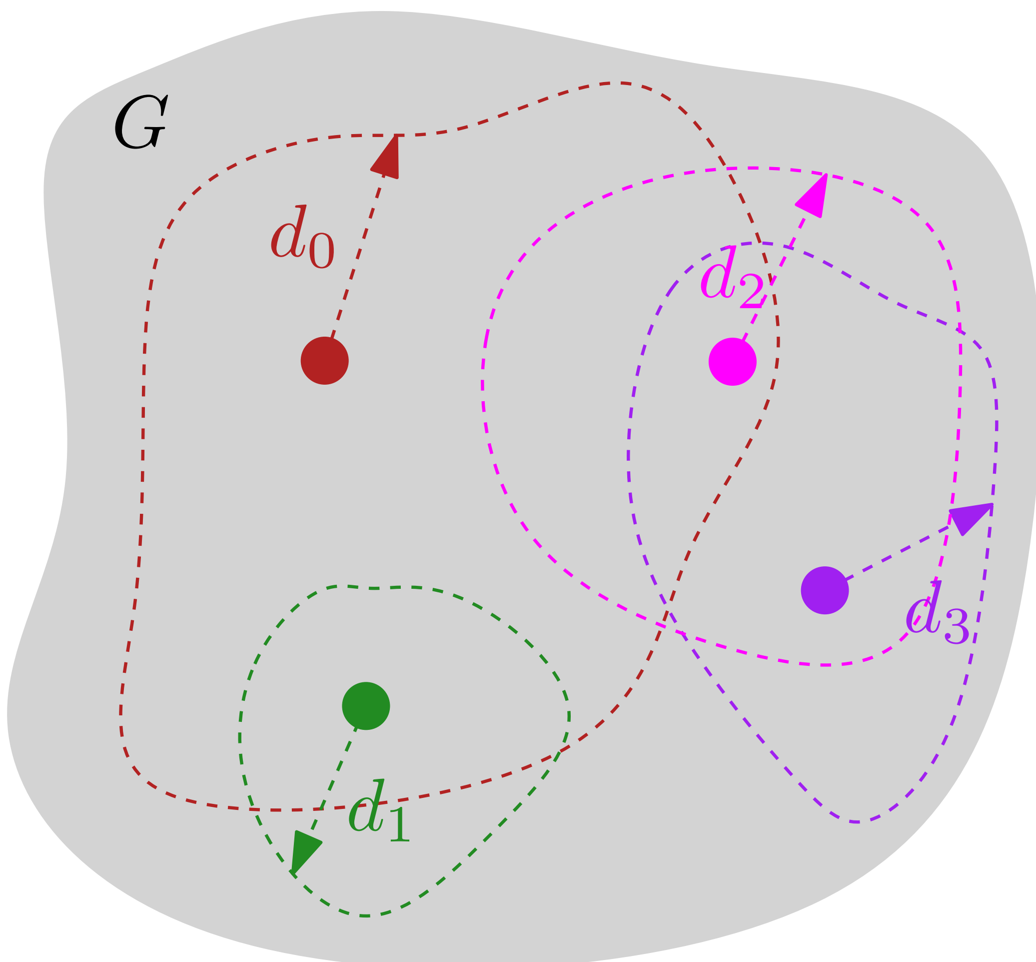

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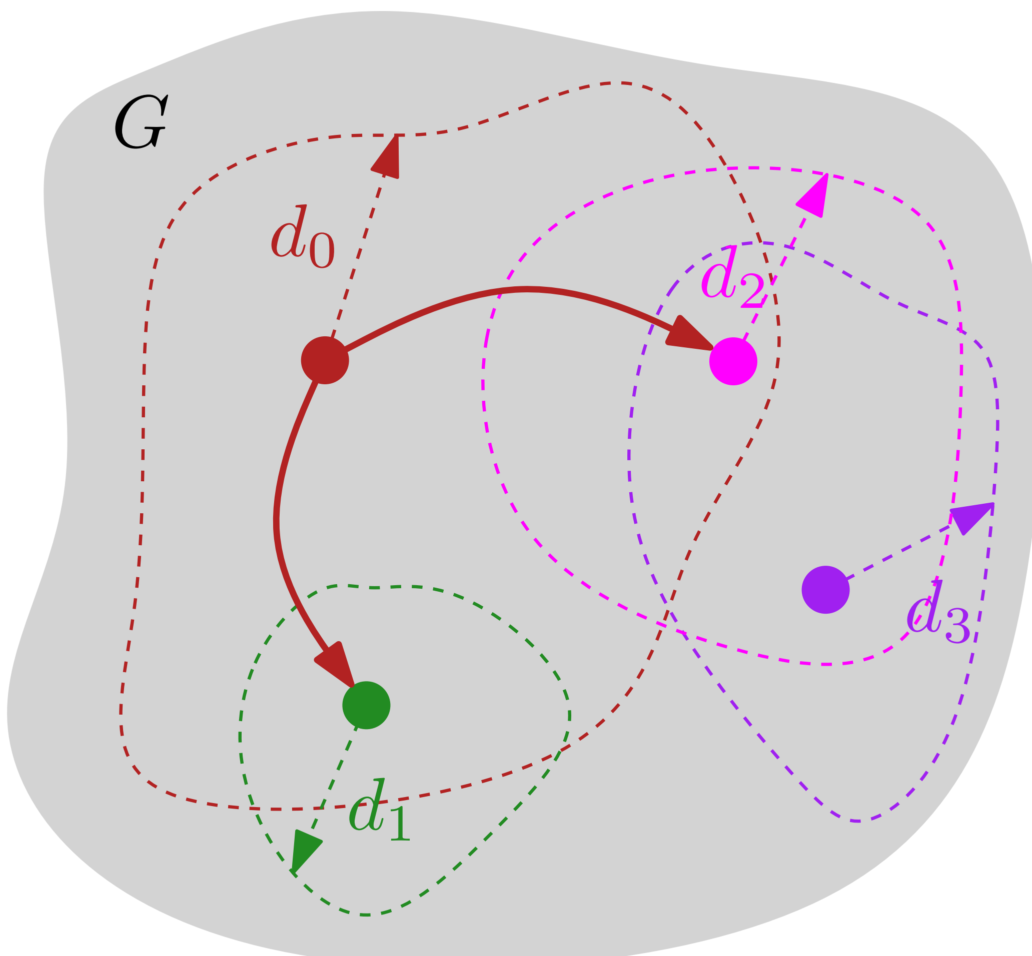

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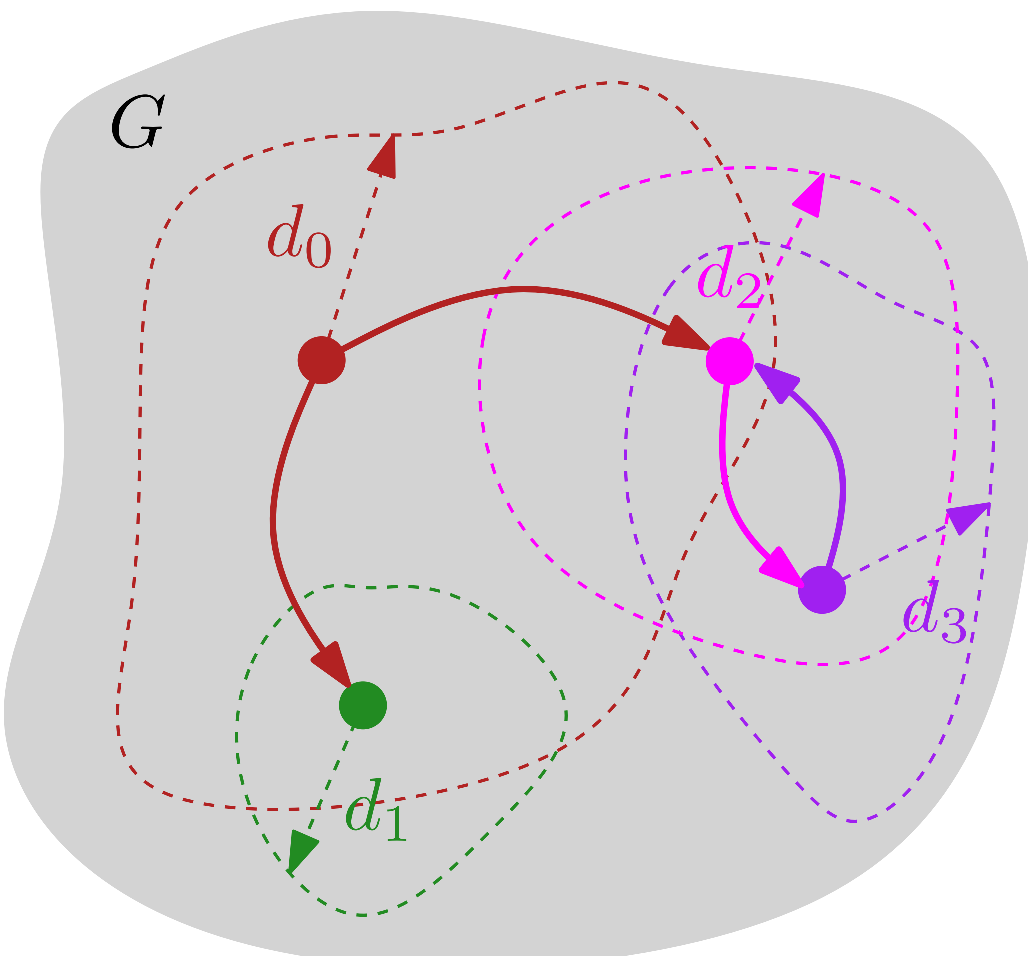

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● and ● interact and
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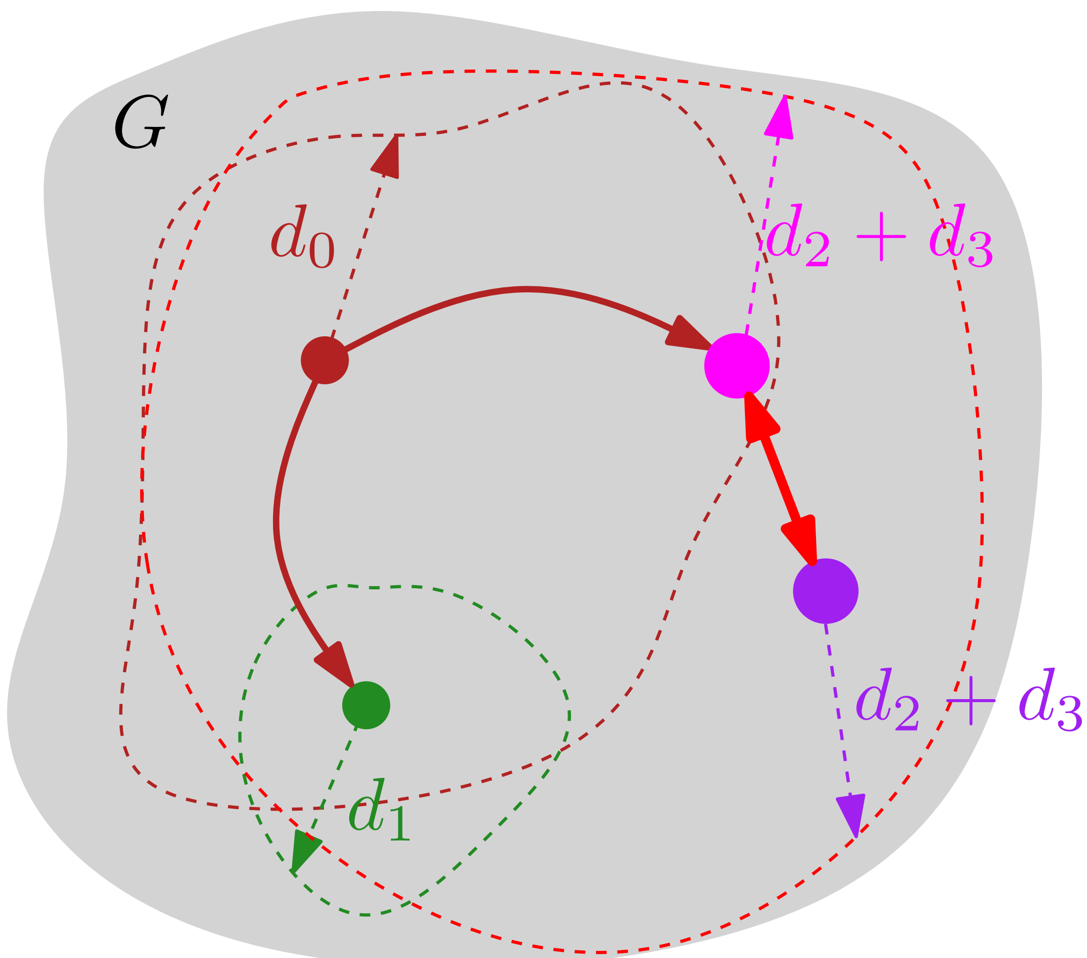

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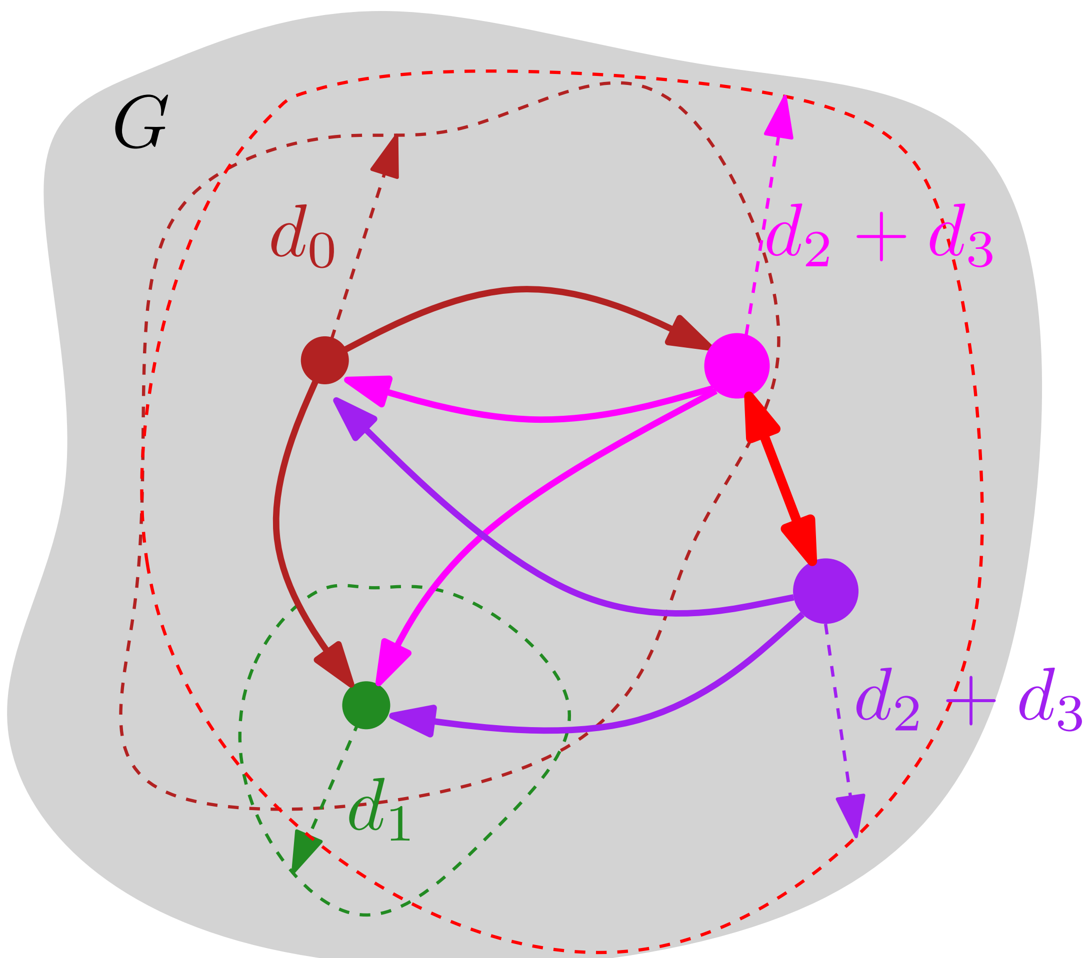

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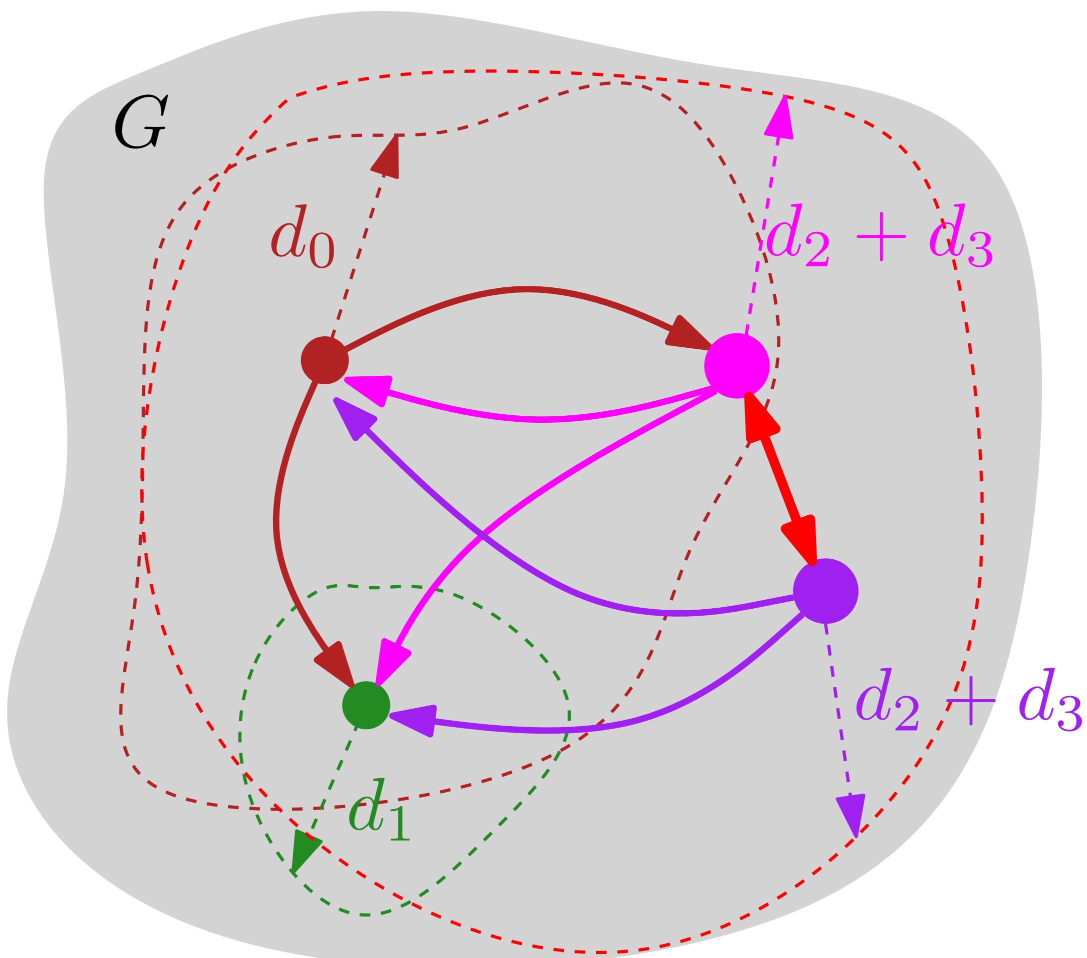

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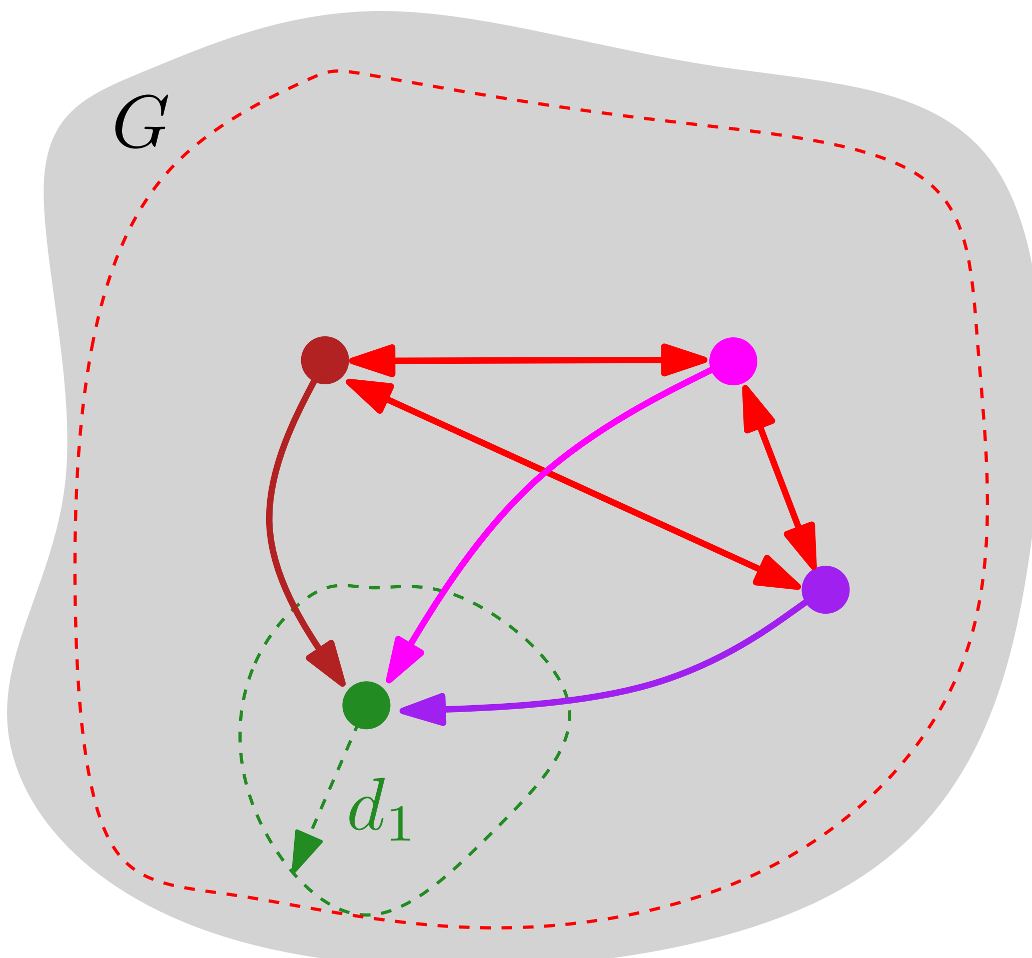

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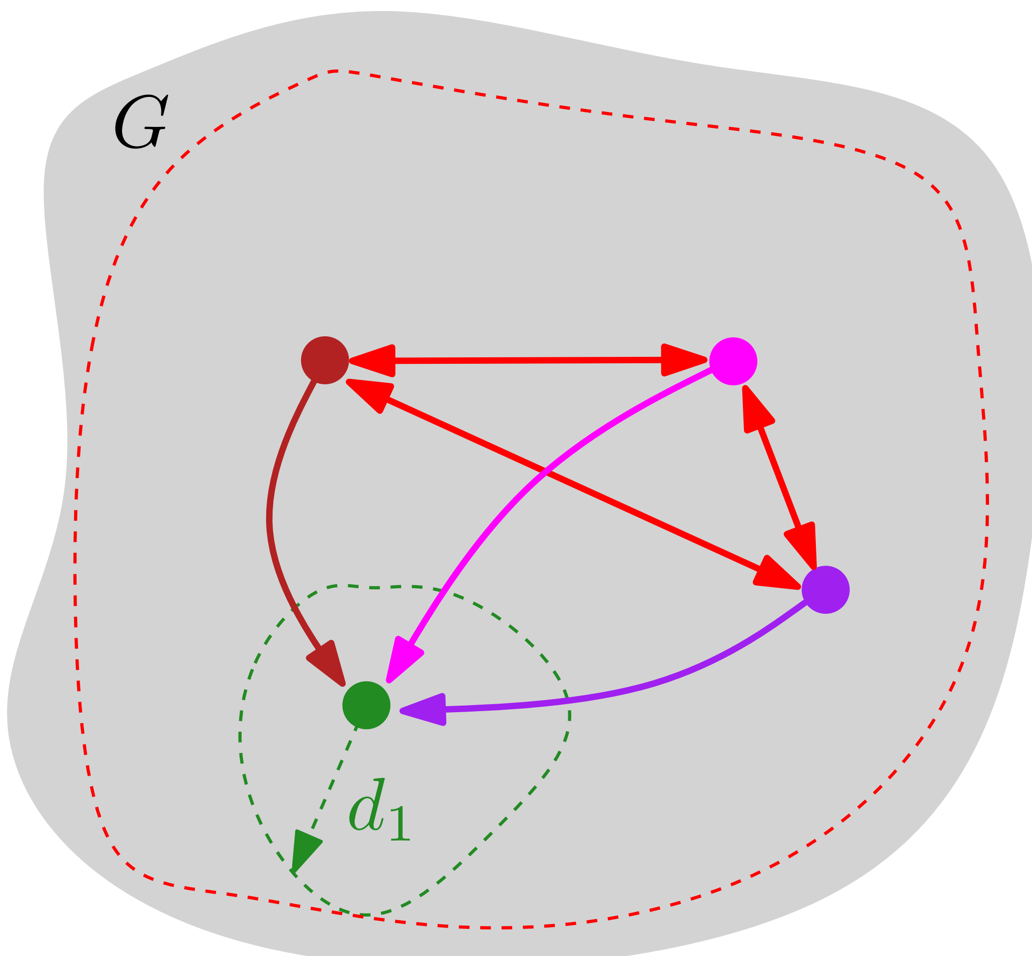

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⟶ Now ● and ● can reach ● or ● !

⟶ ● , ● and ● are all in interaction.

⟶ ● still cannot reach the other 3 vertices to interact.

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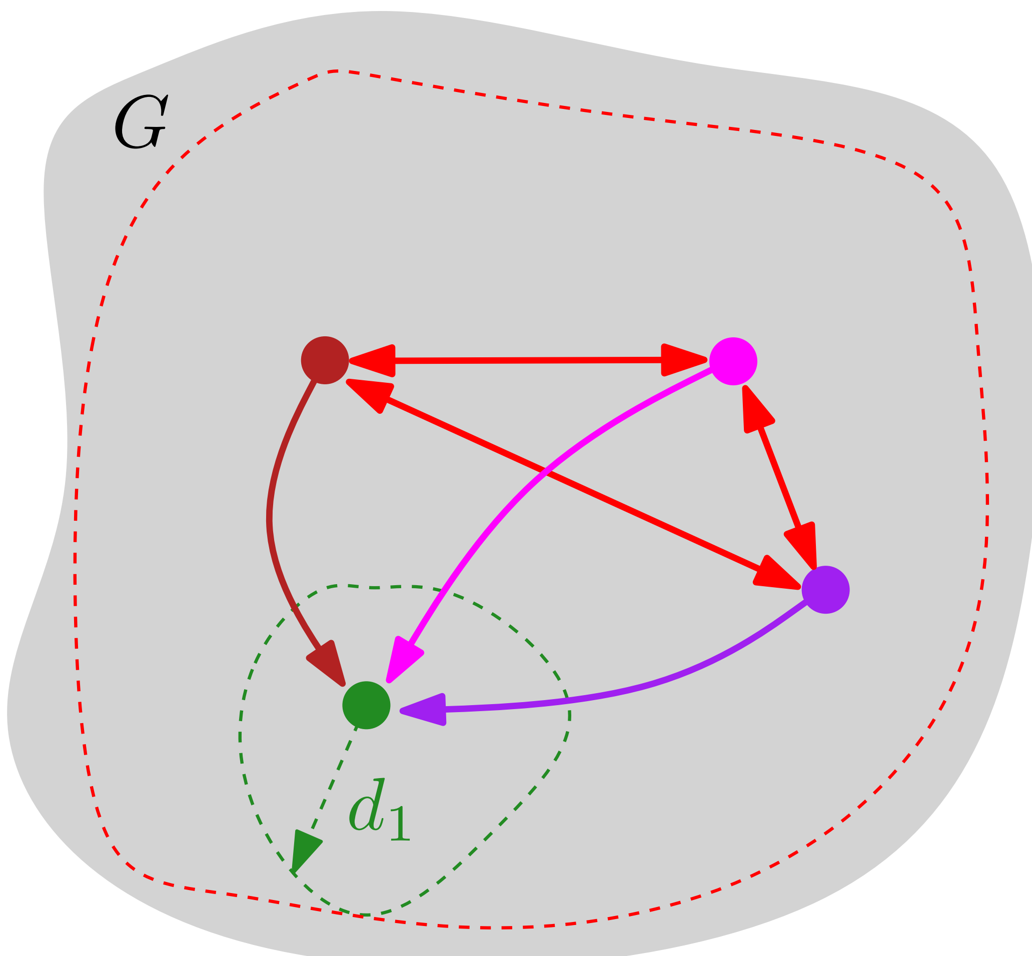
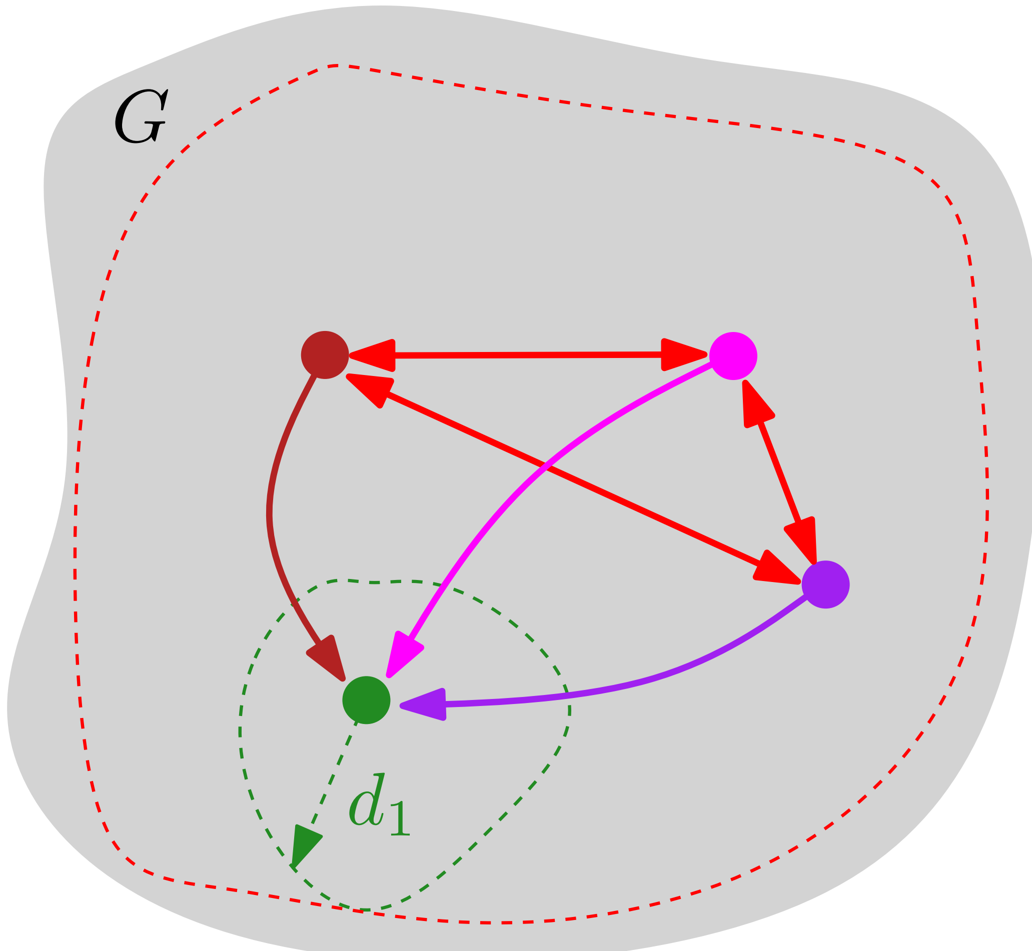


Questions:

- Can we recursively group vertices in classes such that for any two different classes A and B we have:
  $$d(A,B) > \min\{deg(A); deg(B)\}\,?$$
- Is all this hand waving valid?

Heuristics for the contact process



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Cumulative Merging

Cumulative Merging: Admissible partitions

Consider a weighted graph $G = (V, E, r)$ with $r : V \to [0, \infty]$.

Definition

a partition $\mathcal{P}$ of V is **admissible** *iff* $\forall A \neq B \in \mathcal{P}$:

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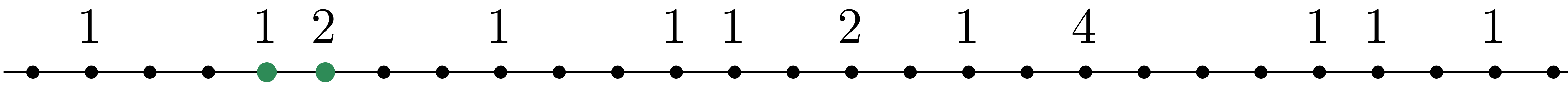
Definition

$$\mathscr{C}(G, r) := \bigcap_{\text{admissible } \mathcal{P}} \mathcal{P} \qquad \text{(finest admissible partition)}$$

Cumulative Merging: Example and basic properties



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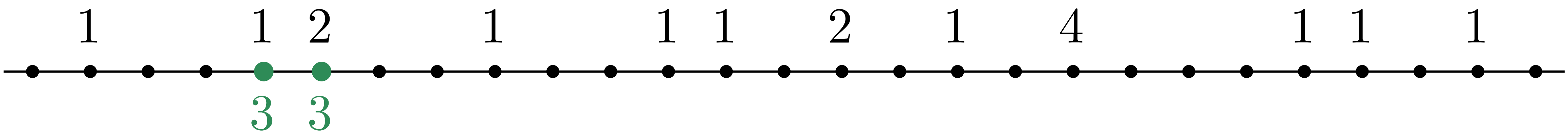

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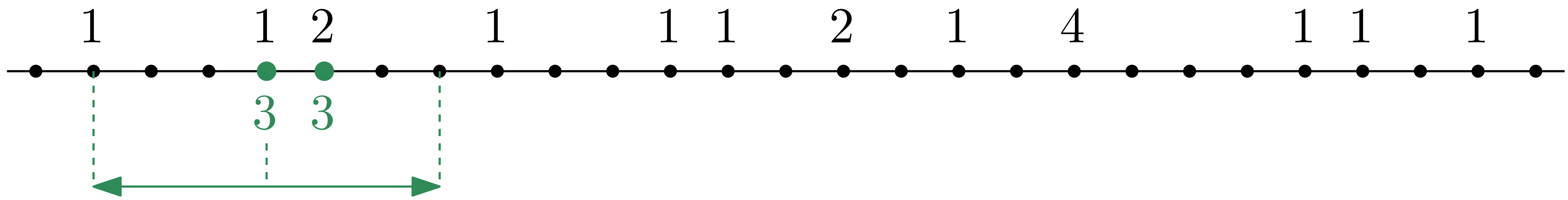

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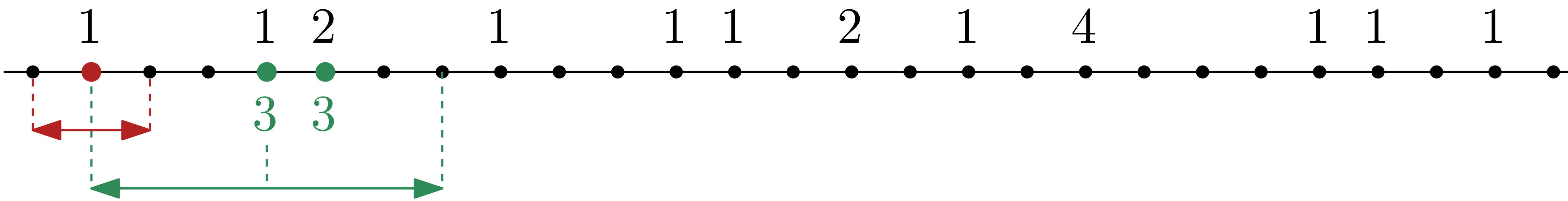

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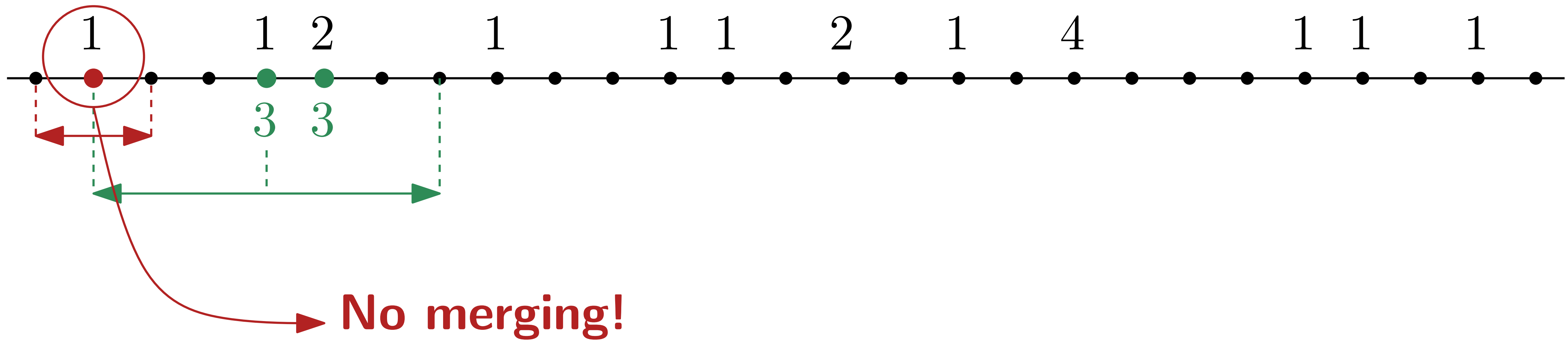

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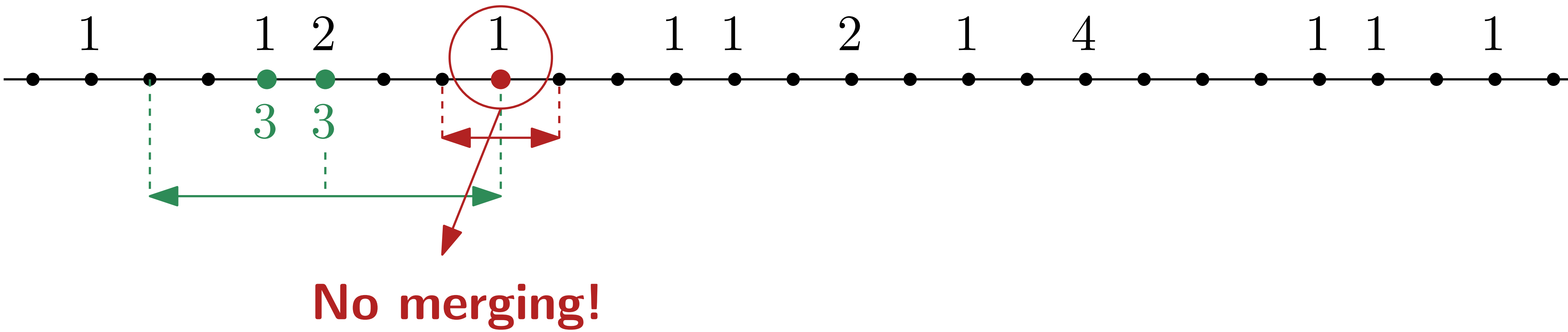

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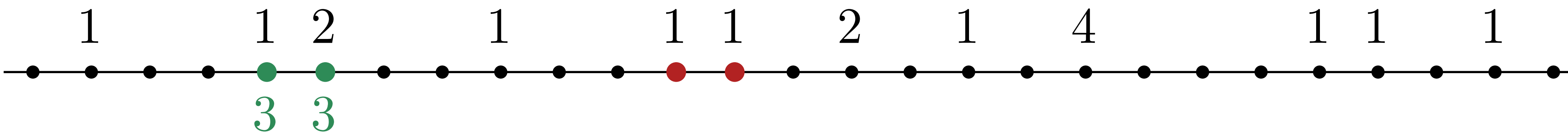

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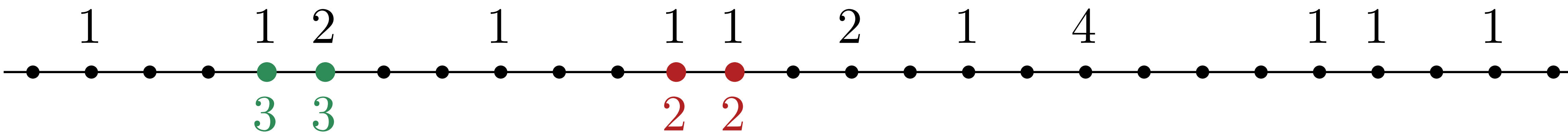

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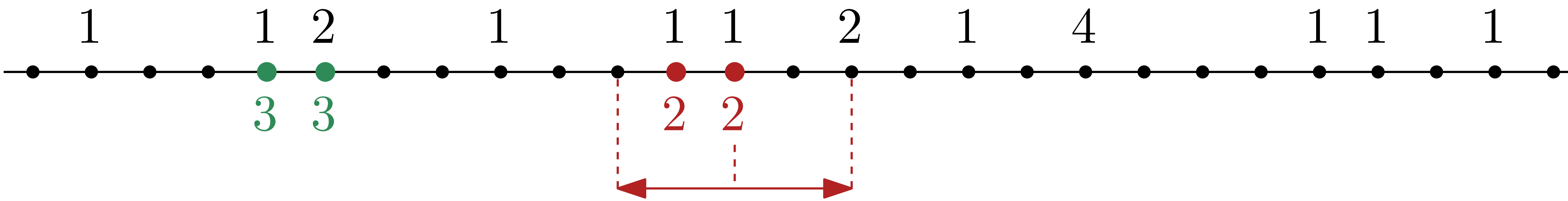

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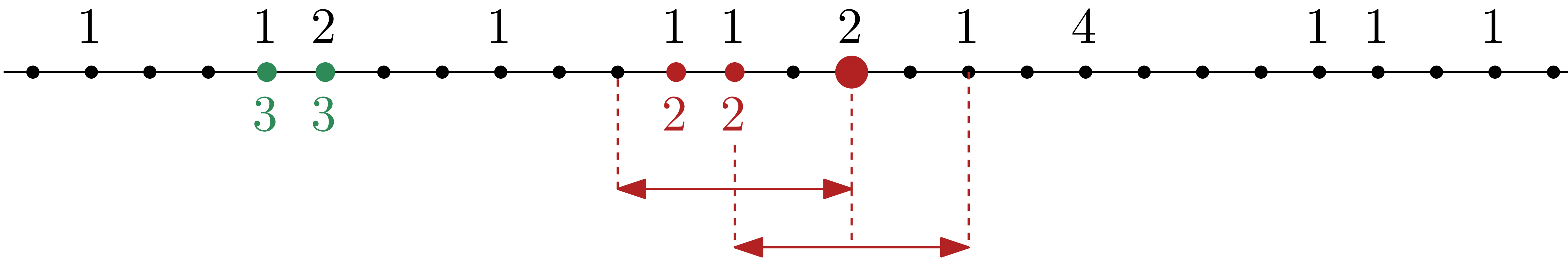

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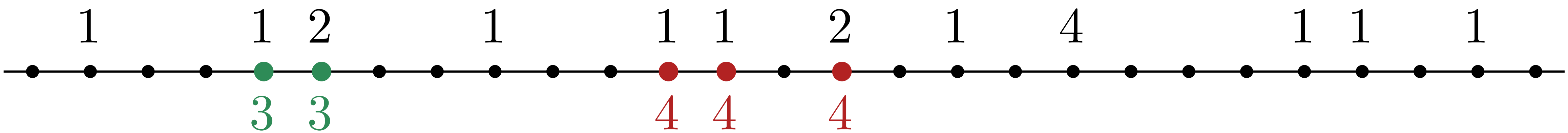

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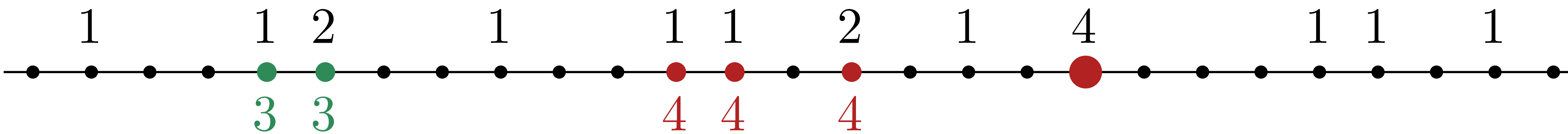

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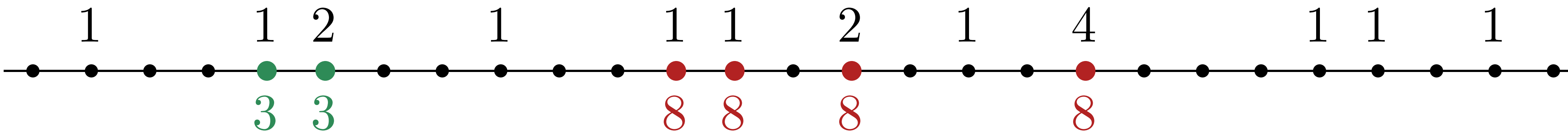

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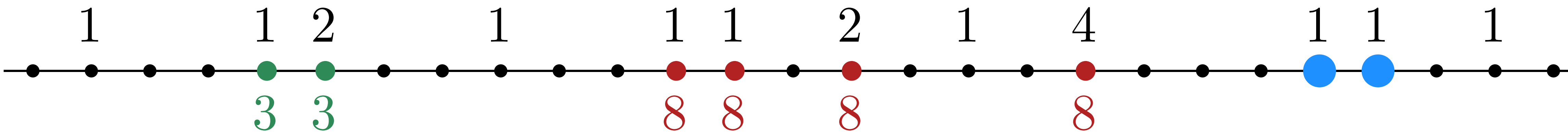

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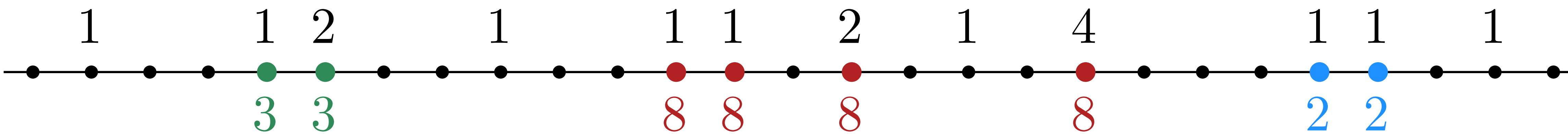

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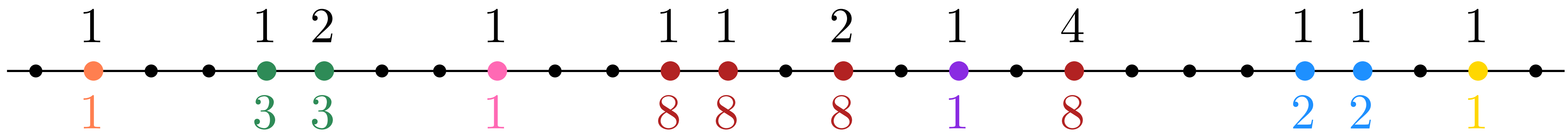

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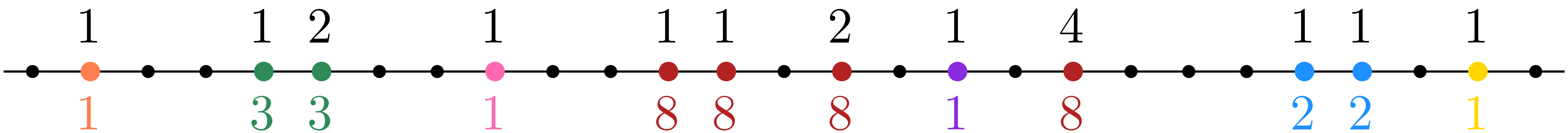

Proposition

- Clusters in $\mathscr{C}$ are not necessarily connected sets!
- If $r(x) < 1$, then $\{x\} \in \mathscr{C}$.
- If $\mathscr{C}$ has an infinite cluster, it has infinite weight and is unique.
- For any $C \in \mathscr{C}$, one has $|C| \leq \max\{1, r(C)\}$.

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Theorem:

1. CMP on $\mathbb{Z}^d$: $p_c \in (0,1)$.
2. CMP on $\mathbb{Z}^d$: if $E\left[Z^\beta\right] < \infty$ for $\beta > (4d)^2$, then $\lambda_c \in (0,\infty)$.
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Proofs: Multiscale analysis

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Thank you!