Traveling waves dans le modèle de Kuramoto quenched

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Travail en commun avec Christophe Poquet (Lyon 1)
[Giacomin, L., Poquet, 2014]
[L., Poquet, 2015, arXiv :1505.00497]

Outline

The stochastic Kuramoto model

Consider the system of N mean-field interacting diffusions

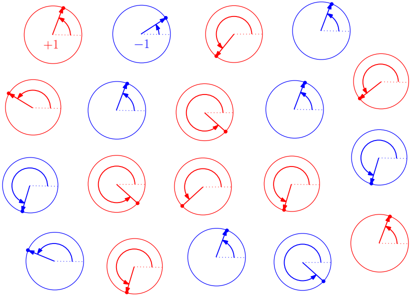
$$\mathrm{d}\theta_{i,t} = \omega_i \mathrm{d}t + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_{j,t} - \theta_{i,t})\mathrm{d}t + \mathrm{d}B_{i,t}, \quad i = 1, \cdots, N,$$

- $\theta_i \in \mathbb{S} := \mathbb{R}/2\pi$ (each particle is an angle),
- $K > 0$: interaction strength,
- $\{B_i\}_i$: i.i.d. Brownian motions,
- $\{\omega_i\}_i$: i.i.d. $\sim \lambda$ (local frequency for each particle, random environment).

Introduced in '70 to describe synchronisation phenomena.

$$\mathrm{d}\theta_{i,t} = \omega_i \mathrm{d}t + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_{j,t} - \theta_{i,t})\mathrm{d}t + \mathrm{d}B_{i,t}, \quad i = 1, \cdots, N,$$

If for example, the frequencies (ω_i) are chosen along $\lambda = \frac{1}{2}(\delta_{-1} + \delta_{+1})$,



The Kuramoto model is invariant by rotation : if $\{\theta_j(t)\}_{j=1\ldots N}$ is a solution, so is $\{\theta_j(t) + \psi\}_{j=1\ldots N}$, for all $\psi \in \mathbb{S}$.

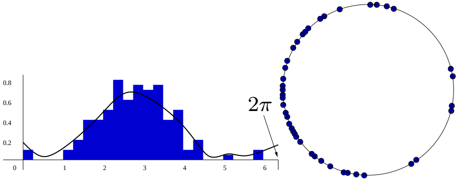
A particular case of mean-field interacting diffusions

The interacting kernel is a function of the (disordered) empirical measure

$$\mu_{N,t} = \frac{1}{N} \sum_{i=1}^{N} \delta_{(\theta_{i,t}, \omega_i)}, \; N \geqslant 1, \; t > 0,$$

so that the dynamics can be rewritten as

$$\mathrm{d}\theta_{i,t} = \omega_i \mathrm{d}t - K(\sin * \mu_{N,t})(\theta_{i,t}, \omega_i)\mathrm{d}t + \mathrm{d}B_{i,t}, \; i = 1, \cdots, N,$$



The nonlinear process

A good candidate for the limiting process as $N \to \infty$ for each particle is

$$d\bar{\theta}_t = \omega dt + K \int_{\mathbb{S}\times\mathbb{R}} \sin(\bar{\theta}_t - \tilde{\theta}) \mathcal{L}(\bar{\theta}_t)(d\tilde{\theta}, d\omega) dt + dB_t,$$

Using only the Lipchitz continuity of the coefficients, one obtains [Sznitman]

$$\mathbf{E}\left[\sup_{t\in[0,T]} \left|\theta_{i,t} - \bar{\theta}_t\right|^2\right] \leqslant \frac{e^{CT}}{N}.$$

As it is, this result is only useful up to time of order $c\log(N)$!

Rq : propagation of chaos uniform in time possible under supplementary convexity assumptions on the kernels (e.g. [Bolley, Guillin, Malrieu, Gentil]).

General framework for the asymptotics of mean-field systems

Aim : understand the behavior of the empirical measure $\mu_{N,t}$, in large population ($N \to \infty$) and in large time ($t \to \infty$).

- $N \to \infty$ with t fixed(propagation of chaos) : the limiting object solves a nonlinear Fokker-Planck PDE, (quite general) [Sznitman, McKean, Gartner, Méléard, Jourdain, Malrieu, Bolley, Guillin, etc.]...
- ... then $t \to \infty$: stationary solutions of this PDE ? (system dependent), [Giacomin, Poquet, Bertini, Pellegrin, Pakdaman, L.]
- The purpose is to look at the asymptotics of μ_N for

$$(t, N) \sim (\tau\sqrt{N}, N), \quad \text{for } N \to \infty.$$

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Large population limit for bounded times : $\mu_{N,t} \to \mu_{\infty,t}$, for $t \in [0, T]$.

Proposition (McKean-Vlasov equation (L. '11))

When $\mathbb{E}(|\omega|) < +\infty$*, for a.e.* $(\omega_i)_{i \geqslant 1}$*, for any* $T > 0$*,* $(\mu_{N,t})_{t \in [0,T]}$ *converges, as* $N \to \infty$ *to*

$$\mu_{\infty,t}(\mathrm{d}\theta, \mathrm{d}\omega) = p_t(\theta, \omega)\mathrm{d}\theta\lambda(\mathrm{d}\omega),$$

where p_t *solves*

$$\partial_t p_t(\theta, \omega) = \frac{1}{2}\partial_\theta^2 p_t(\theta, \omega) - \partial_\theta \left(p_t(\theta, \omega) \left(\omega - K \int \sin(\cdot) * p_t(\cdot, \tilde{\omega})\lambda(\mathrm{d}\tilde{\omega}) \right) \right).$$

$p_t(\theta, \omega)$: density of particles with θ and frequency ω, when $N = \infty$.
For now, suppose that the random environment is chosen in $\{\pm 1\}$. In this case, write

$$p_t^+(\theta) = p_t(\theta, +1),$$
$$p_t^-(\theta) = p_t(\theta, -1).$$

Outline

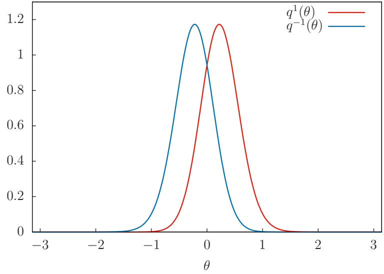
The large population limit

Recall that the random environment is chosen in $\{\pm 1\}$. The stationary solutions of the McKean-Vlasov PDE writes

$$0 = \frac{1}{2}\partial_\theta^2 p^\pm(\theta) - \partial_\theta\left(p^\pm(\theta)\left(-K\sin * \left(\frac{p^+ + p^-}{2}\right)(\theta)\right)\right).$$

- $(q^+, q^-) = (\frac{1}{2\pi}, \frac{1}{2\pi})$ is always a stationary solution (incoherence),
- there exists $K_c > 0$, if $K > K_c$, there exists a whole circle of synchronized stationary solutions

$$M = \{q_\psi(\cdot) = q_0(\cdot - \psi), \psi \in \mathbb{S}\}.$$



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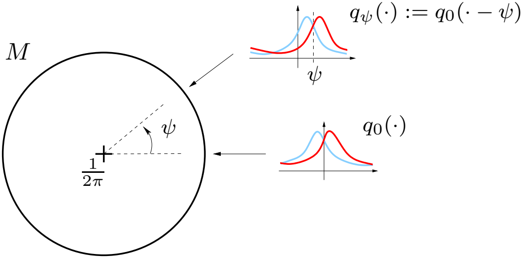


Local stability of the circle M

Let $q = (q^+, q^-) \in M$. Add a small perturbation $u_t = (u_t^+, u_t^-)$, with $\int_{\mathbb{S}} u_t^\pm(\theta)\mathrm{d}\theta = 0$. The evolution of u is given by

$$\partial_t u_t = L_q u_t + R_q(u_t),$$

where

$$L_q u_t^\pm = \frac{1}{2}\partial_\theta^2 u_t^\pm - \partial_\theta \left(\pm\omega_0 u_t^\pm - u_t^\pm K \sin * \left(\frac{q^+ + q^-}{2} \right) - q^\pm K \sin * \left(\frac{u_t^+ + u_t^-}{2} \right) \right),$$

is the linearization of q and R_q is quadratic.

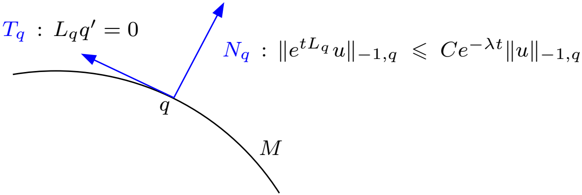
Definition

For $q \in M$, let H_q^{-1} be the dual of H_q^1, weighted Sobolev space endowed with

$$\|u\|_{1,q} := \left(\frac{1}{2} \sum_{\sigma=\pm} \int_{\mathbb{S}} (\partial_\theta u^\sigma(\theta))^2 q^\sigma(\theta) \mathrm{d}\theta \right)^{\frac{1}{2}} .$$

Theorem [Bertini, Giacomin, Pakdaman, 2010], [Giacomin, L., Poquet, 2014] :

For $q \in M$, L_q is self-adjoint in H_q^{-1} and the following decomposition holds : $H_q^{-1} = T_q \oplus M_q$ where



The neutral direction T_q reflects the invariance by rotation of the system.

Idea of proof : perturbing the case without disorder.

1. Without disorder ($\omega_i \equiv 0$),

$$\mathrm{d}\theta_{i,t} = \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_{j,t}-\theta_{i,t})\mathrm{d}t + \mathrm{d}B_{i,t}, \quad i = 1, \cdots, N,$$

the system is reversible with invariant measure $\pi_{N,K}$ (XY model)

$$\pi_{N,K}(\mathrm{d}\theta) \propto \exp\left(\frac{K}{N}\sum_{i,j=1}^{N}\cos(\theta_i - \theta_j)\right)\mathrm{d}\theta$$

The McKean-Vlasov equation can be written in a gradient form :

$$\partial_t q_t(\theta) = \nabla\left[q_t(\theta)\nabla\left(\frac{\delta\mathcal{F}(q_t)}{\delta q_t(\theta)}\right)\right],$$

for

$$\mathcal{F}(q) = \frac{1}{2}\int_{\mathbb{S}} q(\theta)\log q(\theta)\mathrm{d}\theta - \frac{K}{2}\int_{\mathbb{S}^2}\cos(\theta-\theta')q(\theta)q(\theta')\mathrm{d}\theta\mathrm{d}\theta'.$$

[Bertini, Giacomin, Pakdaman, 2010]

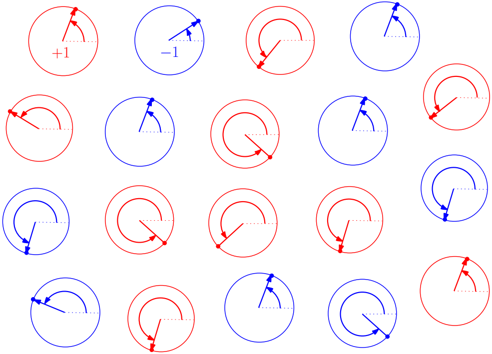
2. This invariant manifold is stable by small perturbation.

Outline

$\mu = \frac{1}{2}\left(\delta_{-1} + \delta_1\right)$.

Symmetric disorder

Suppose that $(\omega_i)_{i \geqslant 1}$ is chosen along $\lambda = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$.



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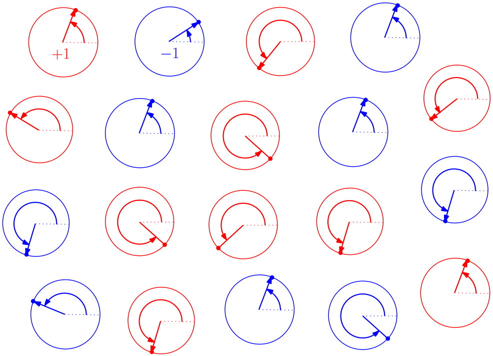

- Here, as $N = \infty$, there is no asymmetry red/blues, so no reasons to rotate in any direction whatsoever : the limit $\mu_{\infty,t}$ gives no information.
- But for N large but finite, fluctuations of the sample $(\omega_1, \dots, \omega_N)$ induce a microscopic asymmetry of order $\sqrt{N}$.
- One expects random traveling waves on a time scale $\sqrt{N}$.

Reformulation of the system

Let $N \geqslant 1$ and N^+ (resp. N^-) the random size of the population with disorder $+1$ (resp. -1). Hence, one can rewrite

$$\mathrm{d}\theta^+_{j,t} = +1 + \frac{K}{N}\left(\sum_{l=1}^{N^+} \sin(\theta^+_{l,t} - \theta^+_{j,t}) + \sum_{l=1}^{N^-} \sin(\theta^-_{l,t} - \theta^+_{j,t})\right) \mathrm{d}t + \mathrm{d}B_{j,t},$$

$$\mathrm{d}\theta^-_{j,t} = -1 + \frac{K}{N}\left(\sum_{l=1}^{N^+} \sin(\theta^+_{l,t} - \theta^-_{j,t}) + \sum_{l=1}^{N^-} \sin(\theta^-_{l,t} - \theta^-_{j,t})\right) \mathrm{d}t + \mathrm{d}B_{j,t}$$

The empirical measure μ_N is identified with (μ_N^+, μ_N^-)

$$\mu^\pm_{N,t} = \frac{1}{N^\pm} \sum_{j=1}^{N^\pm} \delta_{\theta^\pm_{j,t}}.$$

Within this reformulation, the random environnement consists in the random sizes (N^+, N^-).

Quenched traveling waves in long time

Definition

A sequence $(\omega_i)_{i\geqslant 1}$ is *admissible* if for all $\zeta > 0$, there exists N_0 such that for $N \geqslant N_0$, $\max(\left|\xi_N^+\right|, \left|\xi_N^-\right|) \leqslant N^\zeta$, with

$$\xi_N^\pm := N^{1/2}\left(\frac{N^\pm}{N} - \frac{1}{2}\right).$$

Theorem [L., Poquet (2015)] :

Let $\tau_f > 0$, $\psi_0 \in \mathbb{S}$ and an *admissible* sequence $(\omega_i)_{i\geqslant 1}$. If for all $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbf{P}\left(\left\|\mu_{N,0} - q_{\psi_0}\right\|_{-1} \leq \varepsilon\right) = 1,$$

then

$$\lim_{N\to\infty} \mathbf{P}\left(\sup_{\tau\in[0,\tau_f]} \left\|\mu_{N,N^{1/2}\tau} - q_{\psi_0 + b(\xi_N)\tau}\right\|_{-1} \leq \varepsilon\right) = 1,$$

where b is linear.

We want to quantify the proximity of μ_N w.r.t. the circle M. Define

$$\nu_{N,t} := \mu_{N,t} - q_\psi ,$$

where $q_\psi \in M$. Introducing the semigroup $e^{tL_{q_\psi}}$, there exists a mild equation for ν_N in H^{-1} :

$$\nu_{N,t} = e^{tL_{q_\psi}} \nu_{N,0} + \int_0^t e^{(t-s)L_{q_\psi}} \left(D_{q,N} + R_{q,N}(\nu_{N,s})\right) \mathrm{d}s + Z_{N,t},$$

where

- $D_{q,N} = \partial_\theta \left(q_\psi \left\{ \left(\frac{N^+}{N} - \frac{1}{2}\right) (K \sin * q_\psi^+) + \left(\frac{N^-}{N} - \frac{1}{2}\right) (K \sin * q_\psi^-) \right\} \right)$ is the drift induced by the disorder.
- $R_{q,N}(\nu_N)$ is a quadratic term,
- $Z_{N,t}$ is the noise part.

$$\nu_{N,t} = e^{tL_{q_\psi}} \nu_{N,0} + \int_0^t e^{(t-s)L_{q_\psi}} \left(D_{q,N} + R_{q,N}(\nu_{N,s})\right) \mathrm{d}s + Z_{N,t}.$$

There is a competition between

- the drift D_N and the noise Z_N that move μ_N away from the circle M
- the semigroup e^{tL_q} that keeps μ_N close to M along the normal direction N_q,

Hence :

- the dynamics of μ_N is essentially along the neutral direction T_q : one obtains a traveling wave whose speed depends only on the drift D_N.
- Since D_N is a linear functional of $\left(\frac{N^+}{N} - \frac{1}{2}\right)$, this term is of order $\approx \frac{1}{\sqrt{N}}$. We need to wait a time of order $\sqrt{N}$ to see these traveling waves.

The procedure is to consider the dynamics of ν_N on $[0, \sqrt{N}T]$, by discretization on subintervals $[nT, (n+1)T]$, $n = 0, \ldots, \lfloor\sqrt{N}\rfloor$.

Using the semimartingale decomposition, one proves recursively that if $d(\mu_{N,nT}, M) = O(N^{-1/2+2\zeta})$ for a certain n, then with high probability as $N \to \infty$, on the interval $[nT, (n+1)T]$,

- The drift $D_{N,t}$ and the noise $Z_{N,t}$ are of order $O(N^{-1/2+\zeta})$
- the quadratic term R_N is of order $O(N^{-1+4\zeta})$,
- $\nu_{N,t}$ is of order $O(N^{-1/2+2\zeta})$
- One can recursively define the projection q_{ψ_n} of $\mu_{N,nT}$ on M

$$q_{\psi_n} = P_{\psi_n}(\mu_{N,nT}).$$

The process $\nu_{n,t} := \mu_{N,nt} - q_{\psi_n}$, $t \in [0, T]$ is well defined and one has

Proposition (a priori bound on ν_N)

With a probability close to 1 as $N \to \infty$,

$$\sup_{1 \leqslant n \leqslant \lfloor\sqrt{N}\rfloor} \sup_{t\in[0,T]} \left\|\nu_{n,t}\right\|_{-1} \leqslant CN^{-1/2+2\zeta}.$$

- Mean-field model invariant by rotation,
- Stable invariant manifold in the $N = \infty$-limit,
- When $N \to \infty$ and t fixed, the population is balanced : no traveling wave at the level of $\mu_{\infty,t}$.
- Quenched disorder induces traveling waves at the scale of fluctuations for a finite population on a time scale of order $\sqrt{N}$.

Questions :

- more complicated dynamics (FitzHugh-Nagumo) ?
- more general graphs of interactions (Erdös-Renyi, locally tree-like graphs) ? [Delattre, Giacomin, L., 2016]

Merci de votre attention !

E. Luçon and C. Poquet *Long time dynamics and disorder-induced traveling waves in the stochastic Kuramoto model*, arXiv :1505.00497.